Balanced Truncation Approach for Unstable Continuous Systems Based on Mapping

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***Abstract*—In recent years, there have been many studies with different order reduction algorithms to solve model order reduction problems. However, most of the proposed algorithms are mainly applicable to stable linear systems. In practical applications, many problems require order reduction of an unstable continuous system. Therefore, order reduction algorithms need to be able to reduce the order of an unstable continuous system. This paper introduces two balanced truncation algorithms based on mapping applied to unstable continuous systems. By flexibly using the continuous-continuous mapping to transform an unstable continuous system to a stable continuous one and vice versa, the first balanced truncation algorithm can reduce the order of the unstable continuous system. The second balanced truncation algorithm flexibly applies continuous-discrete mapping to convert an unstable continuous system to a stable discrete system and vice versa to help the algorithm reduce the order of the unstable continuous system. Applying two algorithms to reduce the order of the 15th-order unstable system shows that the 5th and 4th-order reduced systems can replace the 15th-order unstable system. The results have demonstrated the correctness of the algorithms and opened the possibility of applying algorithms in practice.**

***Keywords*—model order reduction, balanced truncation algorithm, continuous-continuous mapping, continuous-discontinuous mapping**

I. INTRODUCTION

Since the model reduction problem is proposed, various algorithms have been proposed to deal with the order reduction problem in different approaches. The most popular algorithm is the balanced truncation algorithm proposed by Moore [1]. The ability to preserve the stability of the system of the method in [1] was proven in [2]. The formula to calculate the error limit after the model reduction is performed was determined in [3, 4]. The balanced truncation algorithm is implemented by applying equivalent conditions to the simultaneous diagonalization of the Gramian controllable matrix and Gramian observable matrix. The system dynamic in the open-loop form is observed. By the equivalence of two diagonal matrices, the original system described in the random base system can be converted to the equivalent system. The equivalent system is described in the coordinate system in the balanced internal space. From the balanced space, the low order model can be obtained by eliminating the eigenvalues that have little distribution of the system dynamic. The advantage of the algorithm proposed by Moore [1] is that the error of the model reduction is low. In contrast, this algorithm is only applied to the asymptotic stable linear system. The reason is that the original system needs to be asymptotic stable to determine the Granmian control matrix and Granmian observed matrix. In practice, however, there can be stable high order linear systems [5] and unstable high order linear systems [6–12]. Therefore, order reduction algorithms in general and balanced truncation algorithms, in particular, need to be able to reduce the order of both stable and unstable linear systems.

To apply the balanced truncation algorithm proposed by Moore [1], different approaches have been proposed such as projection-free approximate [13–15], low-rank Gramian approximation [16], LQG balanced method [17], balanced truncation method proposed by Zhou [18, 19], balanced truncation method proposed by Zilochian [18, 20], balanced truncation method applying for the discrete system proposed by Boess [21, 22].

In particular, we are most interested in Zilochian's balanced truncation method [17–20] applied to unstable continuous systems and Boess's balanced truncation method [21, 22] applied to unstable discrete-time systems.

As analyzed above, Moore's balanced truncation algorithm [1] is built based on the controllability Gramian and the observability Gramian of the system. To determine these two Gramians, we need to solve the Lyapunov equations. The condition for the Lyapunov equations to have a solution is that the system is stable. Therefore, when a system is unstable, Lyapunov equations cannot be solved, which means that the balanced truncation algorithm [1] cannot be applied to the unstable system. To address this problem, Zilochian proposed the idea of using a continuous-continuous mapping (displacement of the origin) to convert an unstable continuous system to a stable continuous system [20] so that balanced truncation algorithms can be applied. Performing order reduction of a stable

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continuous system according to the balanced truncation algorithm, we obtain a reduced-order stable continuous system. Finally, an inverse mapping (inverse origin displacement) is performed to transform the reduced-order stable continuous system to the reduced-order unstable continuous system, the same as the original unstable continuous system. Thus, by flexibly using the continuous-continuous mapping to convert a stable continuous system into an unstable continuous system and vice versa, Zilochian's algorithm [20] can reduce the order of unstable systems according to the balanced truncation algorithm.

Applying the balanced truncation algorithm to an unstable discrete-time system has the same problem as an unstable continuous system, i.e., Lyapunov equations cannot be solved. Therefore, Boess [21, 22] proposed a discrete-discrete mapping (origin displacement) to convert an unstable discrete-time system to a stable discrete-time system to apply the balanced truncation algorithm. Then, by reducing the order of the stable discrete-time system according to the balanced truncation algorithm, we get the reduced-order stable discrete-time system. Finally, an inverse mapping (inverse origin displacement) is performed to transform the stable continuous reduced-order system to the unstable continuous reduced-order system, the same as the original unstable continuous system. Thus, similar to Zilochian's algorithm [20], Boess's algorithm [21, 22] also flexibly uses discrete-discrete mapping to help the balanced truncation algorithm reduce the order of the unstable discrete-time system.

Meanwhile, in mathematics, there is a continuous-discrete mapping, which is often applied to convert a continuous system to a discrete-time system and vice versa in some electrical problems. However, according to the original definition of this mapping, the properties of the system after performing the continuous-discrete mapping are still the same, i.e., if the continuous system is stable, the discrete-time system is also stable, and if the continuous system is unstable, the discrete-time system is also unstable. From the continuous-discrete mapping and the idea of two algorithms (Zilochian [20], Boess [21, 22]), Minh *et al.* developed a new continuous-discrete mapping to perform two tasks simultaneously: converting an unstable continuous system into an unstable discrete system, and converting an unstable discrete system into a stable discrete system [23]. Thus, an unstable continuous system after performing the new continuous-discrete mapping will be eligible to apply the balanced truncation algorithm. By flexibly using continuous-discrete mapping, Minh *et al.* has made the balanced truncation algorithm applicable to unstable continuous systems [23]. From the ideas of two above algorithms and the continuous-discrete mapping, this paper introduces in detail Zilochian's balanced truncation algorithm [20] and the balanced truncation algorithm based on continuous-discrete mapping [23]. Illustrative examples are given to verify the efficiency of the algorithms.

The layout of the paper is presented as follows. Section II introduces new concepts of a stable continuous system and a stable discrete system. This section also introduces a balanced truncation algorithm based on continuous-continuous mapping, error evaluation, and complete proof. Then, a balanced truncation algorithm based on the continuous-discrete mapping is presented in Section III. Illustrative examples are given in part IV. Finally, the conclusion is drawn in Section V.

II. Balanced Truncation Algorithm Based on the Continuous-Discrete Mapping

A. The Continuous β-Stable System

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}_c x(t) + \mathbf{B}_c \mathbf{u}(t)\\ y(t) &= \mathbf{C}_c x(t) + \mathbf{D}_c \mathbf{u}(t)\end{aligned} \qquad (1)$$

where:

$$\left(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c\right) \in \mathbb{R}^{nxn} \times \mathbb{R}^{n\mathrm{xm}} \times \mathbb{R}^{\mathrm{kx}n} \times \mathbb{R}^{\mathrm{kxm}}$$

$$x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ y(t) \in \mathbb{R}^k.$$

Take $\mathbf{G}_c(s) := \mathbf{C}_c\left(s\mathbf{I} - \mathbf{A}_c\right)^{-1}\mathbf{B}_c + \mathbf{D}_c,\ \mathrm{s} \in \mathbb{C}$, is the transfer function form of the system (1).

Definition 1 [23]: The system (1) is called β-stable continuous systems if $\mathrm{real}(\lambda(\mathbf{A}_c)) < \beta,\ \beta \geq 0$. The set of β-stable continuous systems is denoted as C_β. The $H_{\infty,\beta}$ standard of $\mathbf{G}_c(s) \in C_\beta$ is defined as:

$$\left\|\mathbf{G}_c(s)\right\|_{H_{\infty,\beta}} := \sup_{real\left(\lambda(A_c)\right)<\beta} \sigma_{\max}\left(\mathbf{G}_c(s)\right) = \sup_{\omega\in R} \sigma_{\max}\left(\mathbf{G}_c(\beta + j\omega)\right)$$

where $\sigma_{\max}\left(\mathbf{G}_c(s)\right)$ is the largest singular value of $\mathbf{G}_c(s)$.

In the case $\beta = 0$, the system (1) is the asymptotically stable arccording to the definition. Matrix A is the Huzwitz matrix i.e, $\mathrm{real}(\lambda(\mathbf{A})) < 0$. The $H_{\infty,\beta}$ standard of $\mathbf{G}_c(s)$ is similar to the H_∞ standard of $\mathbf{G}_c(s)$.

$$\left\|\mathbf{G}_c(s)\right\|_{H_{\infty,0}} = \left\|\mathbf{G}_c(s)\right\|_{H_\infty} := \sup_{\omega\in R} \sigma_{\max}\left(\mathbf{G}_c(j\omega)\right)$$

Thus, with the definition of the β-stable continuous system, every continuous system is a β-stable continuous system. A stable continuous system, according to the original definition (asymptotically stable continuous system), is just a particular case of the β-stable continuous system.

We can transform a β-stable continuous system $\mathbf{G}_c(s)$ into an asymptotically stable continuous system $\mathbf{G}_\beta(s)$ by the following transformation: $\left(\mathbf{A}_\beta, \mathbf{B}_\beta, \mathbf{C}_\beta, \mathbf{D}_\beta\right) = \left(\mathbf{A}_c - \beta\mathbf{I}, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c\right)$, called as continuous-continuous transformation.

The property of an asymptotically stable continuous system after the transformation is shown in the following theorem:

Theorem 1 [23]: The continuous system represented by (1), consider $\mathbf{G}_\beta(s)$ with the transformation $\left(\mathbf{A}_\beta,\mathbf{B}_\beta,\mathbf{C}_\beta,\mathbf{D}_\beta\right)=\left(\mathbf{A}_c-\beta\mathbf{I},\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)$. The properties of $\mathbf{G}_\beta(s)$ is:

(i) $\mathbf{G}_\beta(s)$ is asymptotically stable

(ii) The H_∞ standard of $\mathbf{G}_\beta(s)$ is similar to the $H_{\infty,\beta}$ standard of $\mathbf{G}_c(s)$

$$\left\|\mathbf{G}_\beta(s)\right\|_{H_\infty}=\left\|\mathbf{G}_c(s)\right\|_{H_{\infty,\beta}}.$$

Proof [23]:

(i) From $\mathbf{G}_c(s)\in C_\beta$, we have $\text{real}(\lambda(\mathbf{A}_c))<\beta$ and $\text{real}(\lambda(\mathbf{A}_c-\beta\mathbf{I}))<0$

(ii) We see:

$$\begin{aligned}\mathbf{G}_\beta(j\omega)&=\mathbf{C}_\beta\left(j\omega\mathbf{I}-\mathbf{A}_\beta\right)^{-1}\mathbf{B}_\beta+\mathbf{D}_\beta\\&=\mathbf{C}_c\left(j\omega\mathbf{I}-\mathbf{A}_c+\beta\mathbf{I}\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\&=\mathbf{C}_c\left(\left(\beta+j\omega\right)j\omega\mathbf{I}-\mathbf{A}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\&=\mathbf{G}_c\left(\beta+j\omega\right),\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\mathbf{G}_\beta\right\|_{H_\infty}&=\sup_{\omega\in R}\sigma_{\max}\left(\mathbf{G}_\beta(j\omega)\right)\\&=\sup_{\omega\in R}\sigma_{\max}\left(\mathbf{G}_c(\beta+j\omega)\right)=\left\|\mathbf{G}_c\right\|_{H_{\infty,\beta}}.\end{aligned}$$

Thus, from Theorem 1, we can evaluate the norm of β-stable continuous system through the norm of an asymptotically stable continuous system.

B. Balanced Truncation Algorithm Based on the Continuous-Discrete Mapping

Applying the results of Section II.-B. and study in [20], we introduce a balanced truncation algorithm based on the continuous-continuous mapping for a β-stable continuous system as follows:

Algorithm 1: **Balanced truncation algorithm based on the continuous-continuous mapping for a b-stable continuous system [20]**

Input: The transfer function of the β-stable continuous system as:

$$\mathbf{G}_c(s):=\mathbf{C}_c\left(s\mathbf{I}-\mathbf{A}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c,\ s\in\mathbb{C}_\beta$$

Step 1. Convert the β-stable continuous system $\left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)$ into the asymptotically stable continuous system $\mathbf{G}_\beta$ by the 2 steps:

Step 1.1: Determine the pole θ (the most unstable pole of system). Take $\beta=\text{real}(\theta)+\delta$, where $\delta\in\mathbb{R}$ is arbitrary small and $\delta>0$

Step 1.2: Convert the system $\left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)$ into the asymptotically stable continuous system $\mathbf{G}_\beta$ arccording to:

$$\begin{aligned}\mathbf{A}_\beta&=\mathbf{A}_c-\beta\mathbf{I}.\\\mathbf{B}_\beta&=\mathbf{B}_c,\\\mathbf{C}_\beta&=\mathbf{C}_c,\\\mathbf{D}_\beta&=\mathbf{D}_c.\end{aligned}$$

Step 2. Convert the asymptotically stable continuous system $\mathbf{G}_\beta$ into the balanced equivalent system $\left(\hat{\mathbf{A}}_\beta,\hat{\mathbf{B}}_\beta,\hat{\mathbf{C}}_\beta,\hat{\mathbf{D}}_\beta\right)$ by the following steps:

Step 2.1: Calculate Gramian observable matrix $\mathbf{Q}_\beta$ and Gramian controllable matrix $\mathbf{P}_\beta$ of the system $\left(\mathbf{A}_\beta,\mathbf{B}_\beta,\mathbf{C}_\beta,\mathbf{D}_\beta\right)$ by solving two Lyapunov equations:

$$\begin{aligned}\mathbf{A}_\beta\mathbf{P}_\beta+\mathbf{P}_\beta\mathbf{A}_\beta^{\mathrm{T}}&=-\mathbf{B}_\beta\mathbf{B}_\beta^{\mathrm{T}},\\\mathbf{A}_\beta^{\mathrm{T}}\mathbf{Q}_\beta+\mathbf{Q}_\beta\mathbf{A}_\beta&=-\mathbf{C}_\beta^{\mathrm{T}}\mathbf{C}_\beta.\end{aligned}$$

Step 2.2: Calculate the Cholesky analysis of the matrix $\mathbf{P}_\beta=\mathbf{R}_{\beta p}\mathbf{R}_{\beta p}^{\mathrm{T}}$, where $\mathbf{R}_{\beta p}$ is the upper triangular matrix

Step 2.3: Calculate the Cholesky analysis of the matrix $\mathbf{Q}_\beta=\mathbf{R}_{\beta o}\mathbf{R}_{\beta o}^{\mathrm{T}}$, where $\mathbf{R}_{\beta o}$ is the upper triangular matrix

Remark: It is possible to calculate the Cholesky decomposition of controllability Gramian and observability Gramian matrices from the system parameters $\left(\mathbf{A}_\beta,\mathbf{B}_\beta,\mathbf{C}_\beta,\mathbf{D}_\beta\right)$ without going through the steps of calculating these two matrices (Step 2.1) [24–26].

Step 2.4: Calculate the SVD analysis of matrix $\mathbf{R}_{\beta o}\mathbf{R}_{\beta p}^{\mathrm{T}}=\mathbf{U}_\beta\mathbf{\Lambda}\mathbf{V}_\beta^{\mathrm{T}}$

Step 2.5: Calculate the non-singular matrix $\mathbf{T}_\beta$ $\mathbf{T}_\beta^{-1}=\mathbf{R}_{\beta p}\mathbf{V}_\beta\mathbf{\Lambda}_\beta^{-1/2}$.

Step 2.6:

Calculate $\left(\hat{\mathbf{A}}_\beta,\ \hat{\mathbf{B}}_\beta,\ \hat{\mathbf{C}}_\beta\right)=\left(\mathbf{T}_\beta^{-1}\mathbf{A}_\beta\mathbf{T}_\beta,\ \mathbf{T}_\beta^{-1}\mathbf{B}_\beta,\ \mathbf{C}_\beta\mathbf{T}_\beta\right)$

Remark: Step 2.1 to step 2.6, related to the algorithm for converting the system to the equilibrium form, are referred to [27].

Step 3: Reduce the balanced equivalent system $\left(\hat{\mathbf{A}}_\beta,\hat{\mathbf{B}}_\beta,\hat{\mathbf{C}}_\beta,\hat{\mathbf{D}}_\beta\right)$. We get the asymptotically stable continuous reduced system $\hat{\mathbf{G}}_{1\beta}$ by the following steps:

Step 3.1: Select the reduced-order r so that *r* < *n*.

Step 3.2: Represent $\left(\hat{\mathbf{A}}_\beta,\hat{\mathbf{B}}_\beta,\hat{\mathbf{C}}_\beta,\hat{\mathbf{D}}_\beta\right)$ in the form:

$$\begin{aligned}\hat{\mathbf{A}}_\beta&=\begin{bmatrix}\hat{\mathbf{A}}_{11\beta}&\hat{\mathbf{A}}_{12\beta}\\\hat{\mathbf{A}}_{21\beta}&\hat{\mathbf{A}}_{22\beta}\end{bmatrix},\hat{\mathbf{B}}_\beta=\begin{bmatrix}\hat{\mathbf{B}}_{1\beta}\\\hat{\mathbf{B}}_{2\beta}\end{bmatrix},\\\hat{\mathbf{C}}_\beta&=\begin{bmatrix}\hat{\mathbf{C}}_{1\beta}&\hat{\mathbf{C}}_{2\beta}\end{bmatrix},\hat{\mathbf{D}}_\beta=\hat{\mathbf{D}}_\beta.\end{aligned}$$

where $\hat{\mathbf{A}}_{11\beta}\in\mathbb{R}^{rxr}$, $\hat{\mathbf{B}}_{1\beta}\in\mathbb{R}^{rxp}$, $\hat{\mathbf{C}}_{1\beta}\in\mathbb{R}^{qxr}$.

We get the reduced-order asymptotically stable continuous system $\hat{\mathbf{G}}_{1\beta}$. The system is represented in the form of $\left(\hat{\mathbf{A}}_{11\beta},\ \hat{\mathbf{B}}_{1\beta},\ \hat{\mathbf{C}}_{1\beta},\hat{\mathbf{D}}_\beta\right)$

Step 4. Convert the reduced-order asymptotically stable continuous system system $\hat{\mathbf{G}}_{1\beta}$ into the reduced-order β-stable continuous system $\hat{\mathbf{G}}_1(s)$ by following equations:

$$\begin{aligned}\hat{\mathbf{A}}_1 &= \hat{\mathbf{A}}_{11\beta} + \beta\mathbf{I},\\ \hat{\mathbf{B}}_1 &= \hat{\mathbf{B}}_{1\beta},\\ \hat{\mathbf{C}}_1 &= \hat{\mathbf{C}}_{1\beta}.\\ \hat{\mathbf{D}}_1 &= \hat{\mathbf{D}}_{\beta}\end{aligned}$$

Output: The reduced-order unstable continuous system:

$$\hat{\mathbf{G}}_1(s) = \hat{\mathbf{C}}_1\left(s\mathbf{I} - \hat{\mathbf{A}}_1\right)^{-1}\hat{\mathbf{B}}_1 + \hat{\mathbf{D}}_1$$

To better understand steps of Algorithm 1, the algorithm is represented according to the following Fig. 1:

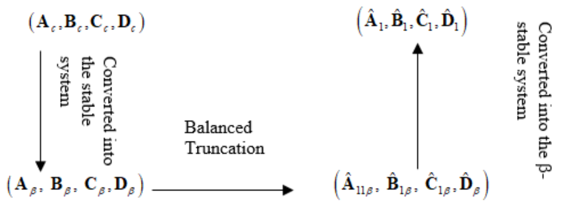


Figure 1. Structure diagram of algorithm 1.

The evaluation of the reduced-order error of the asymptotically stable continuous system according to the balanced truncation algorithm is usually based on the difference between the norm of the original system and the norm of the reduced-order system [1–3]. Using the results of Theorem 1, we can also evaluate the reduced-order error of a β-stable continuous system based on the difference between the norm of the original system and the standard of the asymptotically stable reduced-order system. The evaluation of the reduced-order error of a β-stable continuous system according to Algorithm 1 is presented and proven in Theorem 2 as follows:

Theorem 2 [23]: Let $\mathbf{G}_c(s) \in \mathbb{C}_\beta$ and $\hat{\mathbf{G}}_1(s)$ are the reduced-order system received from the Algorithm 1. We have the formula to calculate the upper bound of the reduced-order error:

$$\left\|\mathbf{G}_c - \hat{\mathbf{G}}_1\right\|_{H_{\infty,\beta}} \le 2\left(\sigma_{r+1} + \ldots + \sigma_n\right),$$

where $\sigma_{r+1} + \ldots + \sigma_n$ are the Hankel singular values of $\mathbf{G}_\beta(s)$

Prove [23]:

Take $\mathbf{E}(s) = \mathbf{G}_c(s) - \hat{\mathbf{G}}_1(s) = \mathbf{C}_e\left(s\mathbf{I} - \mathbf{A}_e\right)^{-1}\mathbf{B}_e + \mathbf{D}_e$.

We get:

$$\mathbf{A}_e = \begin{bmatrix}\mathbf{A}_c & 0\\ 0 & \hat{\mathbf{A}}_1\end{bmatrix},\ \mathbf{B}_e = \begin{bmatrix}\mathbf{B}_c\\ \hat{\mathbf{B}}_1\end{bmatrix},\ \mathbf{C}_e = \begin{bmatrix}\mathbf{C}_c & \hat{\mathbf{C}}_1\end{bmatrix},\ \mathbf{D}_e = 0.$$

From $\mathrm{real}(\lambda(\mathbf{A}_c)) < \beta$, $\mathrm{real}(\lambda(\hat{\mathbf{A}}_1)) < \beta$, We have $\mathbf{E}(s) \in \mathbb{C}_\beta$. Using Theorem 1, we get:

$$\|\mathbf{E}\|_{H_{\infty,\beta}} = \|\mathbf{E}_\beta\|_{H_\infty} = \left\|\mathbf{G}_\beta - \hat{\mathbf{G}}_\beta\right\|_{H_\infty},$$

where $\mathbf{E}_\beta(s) = \mathbf{C}_e\left(s\mathbf{I} - \left(\mathbf{A}_e - \beta I\right)\right)^{-1}\mathbf{B}_e$., the system $\mathbf{G}_\beta$ and $\mathbf{G}_\beta$ are asymptotically stable, the system $\hat{\mathbf{G}}_\beta$ is the reduced-order system of the system $\mathbf{G}_\beta$ received by the balanced truncation algorithm. We get:

$$\left\|\mathbf{G}_\beta - \hat{\mathbf{G}}_\beta\right\|_{H_{\infty,\beta}} \le 2\left(\sigma_{r+1} + \ldots + \sigma_n\right),$$

where $\sigma_{r+1} + \ldots + \sigma_n$ are the Hankel singular values of $\mathbf{G}_\beta(s)$.

III. Balanced Truncation Algorithm Based on the Continuous-Discrete Mapping

A. *The α-Stable Discrete-Time System*

In order to use the continuous-discrete mapping, we introduce some concepts related to stable and unstable discrete-time systems as below:

Consider a discrete linear system represented by:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k)\\ \mathbf{y}(k) &= \mathbf{C}_d\mathbf{x} + \mathbf{D}_k\mathbf{u}(k)\end{aligned} \tag{2}$$

where:

$$\left(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d\right) \in \mathbb{R}^{nxn} \times \mathbb{R}^{nxm} \times \mathbb{R}^{kxn} \times \mathbb{R}^{kxm}$$
$$\mathbf{x}(k) \in \mathbb{R}^n,\ \mathbf{u}(k) \in \mathbb{R}^m,\ \mathbf{y}(k) \in \mathbb{R}^k.$$

The transfer function has the form:

$$\mathbf{G}_d(z) := \mathbf{C}_d\left(z\mathbf{I} - \mathbf{A}_d\right)^{-1}\mathbf{B}_d + \mathbf{D}_d,\ z \in \mathbb{C}$$

Definition 2 [23]: The discrete system (2) is called α-stable discrete-time system if the real part of the pole $|\lambda(\mathbf{A}_d)| < \alpha$, $\alpha \ge 1$. The set of the α- stable discrete-time system is denoted as D_α . The $h_{\infty,\alpha}$ standard of $\mathbf{G}_d(z) \in D_\alpha$ is defined:

$$\begin{aligned}\|\mathbf{G}_d(z)\|_{h_{\infty,\alpha}} &:= \sup_{|z|<\alpha} \sigma_{\max}\left(\mathbf{G}_d(z)\right)\\ &= \sup_{\omega \in R} \sigma_{\max}\left(\mathbf{G}_d(\alpha e^{j\omega})\right),\end{aligned}$$

where $\sigma_{\max}\left(\mathbf{G}_d(z)\right)$ is the largest singular value of $\mathbf{G}_d(z)$.

When $\alpha = 1$, the system (2) is called asymptotically stable discrete-time system. In this case, matrix $\mathbf{A}_d$ is Schur matrix,..., $|\lambda(\mathbf{A}_d)| < 1$. The $h_{\infty,\alpha}$ standard of $\mathbf{G}_d(z)$ is similar to the h∞ standard of $\mathbf{G}_d(z)$ as:

$$\|\mathbf{G}_d(z)\|_{h_{\infty,1}} = \|\mathbf{G}_d(z)\|_{h_\infty} := \sup_{\omega \in R} \sigma_{\max}\left(\mathbf{G}_d(e^{j\omega})\right).$$

Thus, with the definition of an α- stable discrete-time system, every discrete-time system is an α- stable discrete-time system, and an asymptotically stable discrete-time system, according to the original definition, is just a particular case of α- stable discrete-time systems.

B. The Continuous-Discrete Mapping

According to the results of Section II.A and Section III.A, the continuous-discrete mapping is defined by the following definition:

Definition 3 [23]: The mapping

$$\begin{gathered}\Omega_{\beta,\alpha}: C_\beta \to D_\alpha \\ \left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)\to\left(\mathbf{A}_d,\mathbf{B}_d,\mathbf{C}_d,\mathbf{D}_d\right)\end{gathered} \tag{3}$$

where

$$\begin{aligned}\mathbf{A}_d &= \alpha\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right),\\ \mathbf{B}_d &= \sqrt{2\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c,\\ \mathbf{C}_d &= \sqrt{2\alpha}\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1},\\ \mathbf{D}_d &= \mathbf{D}_c+\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}.\end{aligned}$$

$\bar{\mathbf{A}}_c = \mathbf{A}_c - \beta\mathbf{I}$ is called the continuous-discrete mapping. This process converts the β-stable continuous system into the α-stable discrete-time system.

Besides, there is an inverse mapping as:

$$\begin{gathered}\Omega^{-1}_{\beta,\alpha}: D_\alpha \to C_\beta \\ \left(\mathbf{A}_d,\mathbf{B}_d,\mathbf{C}_d,\mathbf{D}_d\right)\to\left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)\end{gathered} \tag{4}$$

where

$$\begin{aligned}\mathbf{A}_c &= \beta\mathbf{I}+\left(\mathbf{I}-\bar{\mathbf{A}}_d\right)^{-1}\left(\bar{\mathbf{A}}_d-\mathbf{I}\right),\\ \mathbf{B}_c &= \sqrt{\frac{2}{\alpha}}\left(\mathbf{I}+\bar{\mathbf{A}}_d\right)^{-1}\mathbf{B}_d,\\ \mathbf{C}_c &= \sqrt{\frac{2}{\alpha}}\mathbf{C}_d\left(\mathbf{I}+\bar{\mathbf{A}}_d\right)^{-1},\\ \mathbf{D}_c &= \mathbf{D}_d-\frac{1}{\alpha}\mathbf{C}_d\left(\mathbf{I}+\bar{\mathbf{A}}_d\right)^{-1}.\end{aligned}$$

$\bar{\mathbf{A}}_d = \dfrac{\mathbf{A}_d}{\alpha}$ is called the inverse continuous-discrete mapping. This process converts the discrete α-stable system into the continuous β-stable system

In the case β = 0 and α = 1, the mapping is called bilinear mapping, i.e., converting an asymptotically stable continuous system to an asymptotically stable discrete-time system and vice versa. Thus, through the continuous-discrete mapping, we can easily convert a β-stable continuous system into an α-stable discrete-time system and vice versa. The property of the continuous-discrete mapping is shown in the following theorem:

Theorem 3 [23]: The arbitrary system $\mathbf{G}_c \in C_\beta, \mathbf{G}_d \in D_\alpha$ with the equivalent representation $\left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)$ and $\left(\mathbf{A}_d,\mathbf{B}_d,\mathbf{C}_d,\mathbf{D}_d\right) = \Omega_{\beta,\alpha}\left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)$. The continuous-discrete mapping remains the following properties:

(i) The continuous-discrete mapping preserves the $H_{\infty,\beta}/h_{\infty,\alpha}$ standard,

$$\left\|\mathbf{G}_c\right\|_{H_{\infty,\beta}} = \left\|\mathbf{G}_d\right\|_{h_{\infty,\alpha}}.$$

(ii) If $\left(\mathbf{A}_c,\mathbf{B}_c,\mathbf{C}_c,\mathbf{D}_c\right)$ is equivalent to the continuous Lyapunov equation:

$$\begin{aligned}&\left(\mathbf{A}_c-\beta\mathbf{I}\right)\mathbf{\Sigma}+\mathbf{\Sigma}\left(\mathbf{A}_c-\beta\mathbf{I}\right)^T+\mathbf{B}_c\mathbf{B}_c^T=0,\\ &\left(\mathbf{A}_c-\beta\mathbf{I}\right)^T\mathbf{\Sigma}+\mathbf{\Sigma}\left(\mathbf{A}_c-\beta\mathbf{I}\right)+\mathbf{C}_c^T\mathbf{C}_c=0.\end{aligned}$$

So $\left(\mathbf{A}_d,\mathbf{B}_d,\mathbf{C}_d,\mathbf{D}_d\right)$ is equivalent to the discontinuous Lyapunov equation

$$\begin{aligned}&\frac{\mathbf{A}_d}{\alpha}\mathbf{\Sigma}\frac{\mathbf{A}_d^T}{\alpha}-\mathbf{\Sigma}+\frac{\mathbf{B}_d}{\sqrt{\alpha}}\frac{\mathbf{B}_d^T}{\sqrt{\alpha}}=0,\\ &\frac{\mathbf{A}_d^T}{\alpha}\mathbf{\Sigma}\frac{\mathbf{A}_d}{\alpha}-\mathbf{\Sigma}+\frac{\mathbf{C}_d^T}{\sqrt{\alpha}}\frac{\mathbf{C}_d}{\sqrt{\alpha}}=0.\end{aligned}$$

(iii) If $\mathbf{G}_{c1}(s),\mathbf{G}_{c2}(s)\in C_\beta$ and $\mathbf{G}_c(s)=\mathbf{G}_{c1}(s)+\mathbf{G}_{c2}(s)$, then

$$\Omega_{\beta,\alpha}\left(\mathbf{G}_c(s)\right)=\Omega_{\beta,\alpha}\left(\mathbf{G}_{c1}(s)\right)+\Omega_{\beta,\alpha}\left(\mathbf{G}_{c2}(s)\right)$$

The inverse of the above assertion also holds.

Proof [23]:

(i) It holds that

$$\begin{aligned}\mathbf{G}_d(z) &= \mathbf{C}_d\left(z\mathbf{I}-\mathbf{A}_d\right)^{-1}\mathbf{B}_d+\mathbf{D}_d\\ &= \sqrt{2\alpha}\mathbf{C}_c\left(z\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(z\mathbf{I}-\alpha\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)\right)^{-1}\\ &\quad \sqrt{2\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c+\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c\\ &= 2\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\frac{z}{\alpha}\mathbf{I}-\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)\right)^{-1}\\ &\quad \left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c+\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c\\ &= 2\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\frac{z}{\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)-\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)\right)^{-1}\mathbf{B}_c\\ &\quad +\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\ &= \mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}2\mathbf{K}^{-1}\mathbf{B}_c+\mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\end{aligned}$$

where $\mathbf{K}=\dfrac{z}{\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)-\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)$.

We have $\mathbf{K}=\left(\dfrac{z}{\alpha}-1\right)\mathbf{I}-\left(\dfrac{z}{\alpha}+1\right)\bar{\mathbf{A}}_c$ and $2\mathbf{I}+\mathbf{K}=\left(\dfrac{z}{\alpha}+1\right)\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)$

Then, we have:

$$\begin{aligned}\mathbf{G}_d(z) &= \mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}(2\mathbf{I}+\mathbf{K})\mathbf{K}^{-1}\mathbf{B}_c+\mathbf{D}_c\\ &= \mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\frac{z}{\alpha}+1\right)\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)\mathbf{K}^{-1}\\ &= \mathbf{C}_c\left(\frac{z}{\alpha}+1\right)\left(\left(\frac{z}{\alpha}-1\right)\mathbf{I}-\left(\frac{z}{\alpha}+1\right)\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\ &= \mathbf{C}_c\left(\frac{z-\alpha}{z+\alpha}\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\ &= \mathbf{C}_c\left(\frac{z-\alpha}{z+\alpha}\mathbf{I}+\beta\mathbf{I}-\mathbf{A}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\ &= \mathbf{C}_c\left(\left(\frac{z-\alpha}{z+\alpha}+\beta\right)\mathbf{I}-\mathbf{A}_c\right)^{-1}\mathbf{B}_c+\mathbf{D}_c\\ &= \mathbf{G}_c\left(\frac{z-\alpha}{z+\alpha}+\beta\right)\\ &= \mathbf{G}_c(s),\end{aligned}$$

where $s = \frac{z-\alpha}{z+\alpha} + \beta$

As $s = \frac{z-\alpha}{z+\alpha} + \beta$ transforming a complex half plane $\text{real}(s) < \beta$ in a circle $\|z\| < \alpha$. We have:

$$\|\mathbf{G}_c\|_{H_{\infty,\beta}} = \sup_{real(s)<\beta} \sigma_{\max}\left(\mathbf{G}_c(s)\right)$$
$$= \sup_{\|z\|<\alpha} \sigma_{\max}\left(\mathbf{G}_d(z)\right) = \|\mathbf{G}_d\|_{h_{\infty,\alpha}}.$$

(ii) By substituting

$$\mathbf{A}_d = \alpha\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right),$$
$$\mathbf{B}_d = \sqrt{2\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c,$$
$$\bar{\mathbf{A}}_c = \mathbf{A}_c - \beta\mathbf{I}.$$

into the discrete Lyapunov.

$$\frac{\mathbf{A}_d}{\alpha}\mathbf{\Sigma}\frac{\mathbf{A}_d^T}{\alpha} - \mathbf{\Sigma} + \frac{\mathbf{B}_d}{\sqrt{\alpha}}\frac{\mathbf{B}_d^T}{\sqrt{\alpha}} = 0$$

We have:

$$\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)\mathbf{\Sigma}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)^T\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-T}$$
$$-\mathbf{\Sigma} + 2\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c\mathbf{B}_c^T\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-T} = 0$$

or

$$\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)\mathbf{\Sigma}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)^T - \left(\mathbf{I}-\bar{\mathbf{A}}_c\right)\mathbf{\Sigma}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^T + \mathbf{B}_c\mathbf{B}_c^T = 0$$

Similarity, $\left(\bar{\mathbf{A}}_c, \mathbf{C}_c\right)$ corresponds to the discrete-time Lyapunov equation

$$\frac{\mathbf{A}_d^T}{\alpha}\mathbf{\Sigma}\frac{\mathbf{A}_d}{\alpha} - \mathbf{\Sigma} + \frac{\mathbf{C}_d^T}{\sqrt{\alpha}}\frac{\mathbf{C}_d}{\sqrt{\alpha}} = 0.$$

(iii) Definition

$$\mathbf{G}_{c1}(s) = \mathbf{C}_{c1}\left(s\mathbf{I}-\mathbf{A}_{c1}\right)\mathbf{B}_{c1} + \mathbf{D}_{c1},$$
$$\mathbf{G}_{c2}(s) = \mathbf{C}_{c2}\left(s\mathbf{I}-\mathbf{A}_{c2}\right)\mathbf{B}_{c2} + \mathbf{D}_{c2}$$

and $\mathbf{G}_{d1}, \mathbf{G}_{d2}, \mathbf{G}_d$ respectively received from $\mathbf{G}_{c1}, \mathbf{G}_{c2}, \mathbf{G}_c$ through mapping $\Omega_{\beta,\alpha}$. $\mathbf{G}_{d1}, \mathbf{G}_{d2}, \mathbf{G}_d$ is performed by $\left(\mathbf{A}_{d1}, \mathbf{B}_{d1}, \mathbf{C}_{d1}, \mathbf{D}_{d1}\right), \left(\mathbf{A}_{d2}, \mathbf{B}_{d2}, \mathbf{C}_{d2}, \mathbf{D}_{d2}\right), \left(\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d\right)$.

We have:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{A}_{c1} & 0 \\ 0 & \mathbf{A}_{c2} \end{bmatrix}, \mathbf{B}_c = \begin{bmatrix} \mathbf{B}_{c1} \\ \mathbf{B}_{c2} \end{bmatrix},$$
$$\mathbf{C}_c = \begin{bmatrix} \mathbf{C}_{c1} & \mathbf{C}_{c2} \end{bmatrix}, \mathbf{D}_c = \mathbf{D}_{c1} + \mathbf{D}_{c2}.$$

and

$$\bar{\mathbf{A}}_c = \mathbf{A}_c - \beta\mathbf{I} = \begin{bmatrix} \mathbf{A}_{c1}-\beta\mathbf{I} & 0 \\ 0 & \mathbf{A}_{c2}-\beta\mathbf{I} \end{bmatrix}$$
$$= \begin{bmatrix} \bar{\mathbf{A}}_{c1} & 0 \\ 0 & \bar{\mathbf{A}}_{c2} \end{bmatrix}$$

By substituting $\bar{\mathbf{A}}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c$ into $\mathbf{A}_d, \mathbf{B}_d, \mathbf{C}_d, \mathbf{D}_d$. We have:

$$\mathbf{A}_d = \alpha\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_c\right)$$
$$= \alpha\begin{bmatrix} \mathbf{I}-\bar{\mathbf{A}}_{c1} & 0 \\ 0 & \mathbf{I}-\bar{\mathbf{A}}_{c2} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{I}+\bar{\mathbf{A}}_{c1} & 0 \\ 0 & \mathbf{I}+\bar{\mathbf{A}}_{c2} \end{bmatrix}$$
$$= \alpha\begin{bmatrix} \left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1} & 0 \\ 0 & \left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{I}-\bar{\mathbf{A}}_{c1} & 0 \\ 0 & \mathbf{I}-\bar{\mathbf{A}}_{c2} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha\left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_{c1}\right) & 0 \\ 0 & \alpha\left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1}\left(\mathbf{I}+\bar{\mathbf{A}}_{c2}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}_{d1} & 0 \\ 0 & \mathbf{A}_{d2} \end{bmatrix},$$

$$\mathbf{B}_d = \sqrt{2\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c$$
$$= \sqrt{2\alpha}\begin{bmatrix} \left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1} & 0 \\ 0 & \left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{B}_{c1} \\ \mathbf{B}_{c2} \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{2\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1}\mathbf{B}_{c1} \\ \sqrt{2\alpha}\left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1}\mathbf{B}_{c2} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{d1} \\ \mathbf{B}_{d2} \end{bmatrix},$$

$$\mathbf{C}_d = \sqrt{2\alpha}\mathbf{C}_c\left(I-\bar{\mathbf{A}}_c\right)^{-1}$$
$$= \sqrt{2\alpha}\begin{bmatrix} \mathbf{C}_{c1} & \mathbf{C}_{c2} \end{bmatrix}\begin{bmatrix} \left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1} & 0 \\ 0 & \left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{2\alpha}\mathbf{C}_{c1}\left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1} & \sqrt{2\alpha}\mathbf{C}_{c2}\left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{C}_{d1} & \mathbf{C}_{d2} \end{bmatrix}$$

$$\mathbf{D}_d = \mathbf{D}_c + \mathbf{C}_c\left(\mathbf{I}-\bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c$$
$$= \mathbf{D}_{c1} + \mathbf{D}_{c2}$$
$$+\begin{bmatrix} \mathbf{C}_{c1} & \mathbf{C}_{c2} \end{bmatrix}\begin{bmatrix} \left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1} & 0 \\ 0 & \left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{B}_{c1} \\ \mathbf{B}_{c2} \end{bmatrix}$$
$$= \mathbf{D}_{c1} + \mathbf{C}_{c1}\left(\mathbf{I}-\bar{\mathbf{A}}_{c1}\right)^{-1}\mathbf{B}_{c1}$$
$$+\mathbf{D}_{c2} + \mathbf{C}_{c2}\left(\mathbf{I}-\bar{\mathbf{A}}_{c2}\right)^{-1}\mathbf{B}_{c2}$$
$$= \mathbf{D}_{d1} + \mathbf{D}_{d2}$$

Therefore

$$\mathbf{G}_d(z) = \mathbf{C}_d\left(z\mathbf{I}-\mathbf{A}_d\right)^{-1}\mathbf{B}_d + \mathbf{D}_d$$
$$= \begin{bmatrix} \mathbf{C}_{d1} & \mathbf{C}_{d2} \end{bmatrix}\begin{bmatrix} z\mathbf{I}-\mathbf{A}_{d1} & 0 \\ 0 & z\mathbf{I}-\mathbf{A}_{d2} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{B}_{d1} \\ \mathbf{B}_{d2} \end{bmatrix}$$
$$+\mathbf{D}_{d1} + \mathbf{D}_{d2}$$
$$= \mathbf{C}_{d1}\left(z\mathbf{I}-\mathbf{A}_{d1}\right)^{-1}\mathbf{B}_{d1} + \mathbf{D}_{d1}$$
$$+\mathbf{C}_{d2}\left(z\mathbf{I}-\mathbf{A}_{d2}\right)^{-1}\mathbf{B}_{d2} + \mathbf{D}_{d2}$$
$$= \mathbf{G}_{d1}(z) + \mathbf{G}_{d2}(z)$$

Thus, if we want to convert a β-stable continuous system into an asymptotically stable discrete system and vice versa, we perform mapping $\Omega_{\beta,1}$ and mapping $\Omega^{-1}_{\beta,1}$, i.e., choose $\beta > 0$ and $\alpha = 1$.

C. Balanced Truncation Algorithm Based on the Continuous-Discrete Mapping

The balanced truncation algorithm [1] can only be applied to stable and asymptotically stable discrete-time systems. Therefore, Boss [21] has built a balanced truncation algorithm that can reduce order for unstable discrete-time systems, called α-stable discrete-time systems, by converting an α-stable discrete-time system to an asymptotically stable discrete-time system (α = 1) to satisfy the condition of using the balanced truncation algorithm. However, this algorithm is only applicable to discrete-time systems. To apply the balanced truncation algorithm of Boss [21] to the unstable continuous system, called β-stable continuous systems, we use the balanced truncation algorithm based on the continuous-discrete mapping. In more detail, the mapping $\Omega_{\beta,1}$ is applied to convert a β-stable continuous system to an asymptotically stable discrete-time system and vice versa to satisfy the condition of applying the balanced truncation algorithm; then the mapping $\Omega^{-1}_{\beta,1}$ is employed to convert an asymptotically stable discrete-time reduced-order system to a β-stable continuous reduced-order system (unstable continuous reduced-order system).

The details of the balanced truncation algorithm based on the continuous-discrete mapping for the β-stable continuous system are as follows:

Algorithm 2: **Balanced Truncation Algorithm based on the continuous-discrete mapping [23]**

Input: The β-stable continuous system $\mathbf{G}_c(s) = \mathbf{C}_c(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c + \mathbf{D}_c \in C_\beta$

Step 1: Convert the β-stable continuous system $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$ into the asymptotically stable discrete system $\mathbf{G}_d \in C$ by the following steps:

Step 1.1 Determine the pole θ which is the most unstable pole of $\mathbf{G}_c(s)$. Take $\beta = \text{real}(\theta) + \delta$, where $\delta \in \mathbb{R}$ is arbitrarily small and $\delta > 0$

Step 1.2: Convert the system $(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$ into the asymptotically stable discrete-time system $\mathbf{G}_d \in C$ according to the system equation:

$$\bar{\mathbf{A}}_c = \mathbf{A}_c - \beta\mathbf{I},$$
$$\mathbf{A}_d = \left(\mathbf{I} - \bar{\mathbf{A}}_c\right)^{-1}\left(\mathbf{I} + \bar{\mathbf{A}}_c\right),$$
$$\mathbf{B}_d = \sqrt{2}\left(\mathbf{I} - \bar{\mathbf{A}}_c\right)^{-1}\mathbf{B}_c,$$
$$\mathbf{C}_d = \sqrt{2}\mathbf{C}_c\left(\mathbf{I} - \bar{\mathbf{A}}_c\right)^{-1},$$
$$\mathbf{D}_d = \mathbf{D}_c + \mathbf{C}_c\left(\mathbf{I} - \bar{\mathbf{A}}_c\right)^{-1}.$$

Step 2. Convert the asymptotically stable discrete-time system $\mathbf{G}_d$ into the equivalent system $(\hat{\mathbf{A}}_d, \hat{\mathbf{B}}_d, \hat{\mathbf{C}}_d, \hat{\mathbf{D}}_d)$ which is also stable.

Step 2.1: Calculate the Gramian observable matrix $\mathbf{Q}_d$ and the Gramian controllable matrix of the system $\mathbf{P}_d$ by solving two Lyapunov.

$$\mathbf{A}_d\mathbf{P}_d + \mathbf{P}_d\mathbf{A}_d^{\mathrm{T}} = -\mathbf{B}_d\mathbf{B}_d^{\mathrm{T}},$$
$$\mathbf{A}_d^{\mathrm{T}}\mathbf{Q}_d + \mathbf{Q}_d\mathbf{A}_d = -\mathbf{C}_d^{\mathrm{T}}\mathbf{C}_d.$$

Step 2.2: The Cholesky analysis of $\mathbf{P}_d = \mathbf{R}_{dp}\mathbf{R}_{dp}^{\mathrm{T}}$, with $\mathbf{R}_{dp}$ is the upper triangular matrix

Step 2.3: The Cholesky analysis of $\mathbf{Q}_d = \mathbf{R}_{do}\mathbf{R}_{do}^{\mathrm{T}}$, with $\mathbf{R}_{do}$ is the upper triangular matrix

Step 2.4: The SVD analysis of $\mathbf{R}_{do}\mathbf{R}_{dp}^{\mathrm{T}} = \mathbf{U}_d\mathbf{\Lambda}_d\mathbf{V}_d^{\mathrm{T}}$

Step 2.5: Calculate the non-singular matrix $\mathbf{T}_d$

$$\mathbf{T}_d^{-1} = \mathbf{R}_{dp}\mathbf{V}_d\mathbf{\Lambda}_d^{-1/2}.$$

Step 2.6:

Calculate

$$\left(\hat{\mathbf{A}}_d,\ \hat{\mathbf{B}}_d,\ \hat{\mathbf{C}}_d, \hat{\mathbf{D}}_d\right) = \left(\mathbf{T}_d^{-1}\mathbf{A}_d\mathbf{T}_d,\ \mathbf{T}_d^{-1}\mathbf{B}_d,\ \mathbf{C}_d\mathbf{T}_d, \mathbf{D}_d\right)$$

Step 3: Truncate the system $(\hat{\mathbf{A}}_d, \hat{\mathbf{B}}_d, \hat{\mathbf{C}}_d, \hat{\mathbf{D}}_d)$ to receive the asymptotically stable discrete-time reduced system $\hat{\mathbf{G}}_d(z)$ by the following steps:

Step 3.1: Choose the order of reduced system r so that $r < n$.

Step 3.2: $(\hat{\mathbf{A}}_d, \hat{\mathbf{B}}_d, \hat{\mathbf{C}}_d, \hat{\mathbf{D}}_d)$ is represented in the form:

$$\hat{\mathbf{A}}_d = \begin{bmatrix} \hat{\mathbf{A}}_{11d} & \hat{\mathbf{A}}_{12d} \\ \hat{\mathbf{A}}_{21d} & \hat{\mathbf{A}}_{22d} \end{bmatrix}, \hat{\mathbf{B}}_d = \begin{bmatrix} \hat{\mathbf{B}}_{1d} \\ \hat{\mathbf{B}}_{2d} \end{bmatrix}, \hat{\mathbf{C}}_d = \begin{bmatrix} \hat{\mathbf{C}}_{1d} & \hat{\mathbf{C}}_{2d} \end{bmatrix}, \hat{\mathbf{D}}_d = \hat{\mathbf{D}}_d,$$

where $\hat{\mathbf{A}}_{11d} \in \mathbb{R}^{rxr}$, $\hat{\mathbf{B}}_{1d} \in \mathbb{R}^{rxp}$, $\hat{\mathbf{C}}_{1d} \in \mathbb{R}^{qxr}$.

We receive the asymptotically stable discrete-time reduced system $\hat{\mathbf{G}}_d$. The reduced system which is stable is represented in the form $(\hat{\mathbf{A}}_{11d}, \hat{\mathbf{B}}_{1d}, \hat{\mathbf{C}}_{1d}, \hat{\mathbf{D}}_d)$

Step 4: Convert the asymptotically stable discrete-time reduced system $\hat{\mathbf{G}}_d(z)$ into the β-stable continuous reduced system $\hat{\mathbf{G}}_2(s)$ by transforming $(\hat{\mathbf{A}}_2, \hat{\mathbf{B}}_2, \hat{\mathbf{C}}_2, \hat{\mathbf{D}}_2) = \Omega^{-1}_{\beta,1}(\hat{\mathbf{A}}_{11d}, \hat{\mathbf{B}}_{1d}, \hat{\mathbf{C}}_{1d}, \hat{\mathbf{D}}_d)$ as:

$$\hat{\mathbf{A}}_2 = \beta\mathbf{I} + \left(\mathbf{I} - \hat{\mathbf{A}}_{11d}\right)^{-1}\left(\hat{\mathbf{A}}_{11d} - \mathbf{I}\right),$$
$$\hat{\mathbf{B}}_2 = \sqrt{2}\left(\mathbf{I} + \hat{\mathbf{A}}_{11d}\right)^{-1}\hat{\mathbf{B}}_{1d},$$
$$\hat{\mathbf{C}}_2 = \sqrt{2}\hat{\mathbf{C}}_{1d}\left(\mathbf{I} + \hat{\mathbf{A}}_{11d}\right)^{-1},$$
$$\hat{\mathbf{D}}_2 = \hat{\mathbf{D}}_d - \hat{\mathbf{C}}_{1d}\left(\mathbf{I} + \hat{\mathbf{A}}_{11d}\right)^{-1},$$

Output: The β-stable continuous reduced system.

$$\hat{\mathbf{G}}_2(s) = \hat{\mathbf{C}}_2\left(s\mathbf{I} - \hat{\mathbf{A}}_2\right)^{-1}\hat{\mathbf{B}}_2 + \hat{\mathbf{D}}_2$$

To better understand the steps of Algorithm 2, we represent Algorithm 2 according to the following figure:

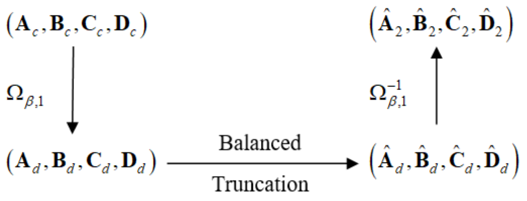


Figure 2. Structure diagram of algorithm 2.

Based on Theorem 1, we evaluate the reduced-order error of a β-stable linear system according to Algorithm 2 as follows:

Theorem 4: [23] Take $\mathbf{G}_c(s) \in C_\beta$ and $\hat{\mathbf{G}}_2(s)$ are the reduced system receive from the algorithm 2. Then we obtain the formula for calculating the upper bound of the order reduction error as follows:

$$\left\| \mathbf{G}_c - \hat{\mathbf{G}}_2 \right\|_{H_{\infty,\beta}} \le 2(\pi_{r+1} + ... + \pi_n),$$

where $\pi_{r+1} + ... + \pi_n$ are the Hankel singular values of $\mathbf{G}_d(z)$.

Prove [23]: By theorem 1, we have:

$$\begin{aligned} \left\| \mathbf{G}_c - \hat{\mathbf{G}}_2 \right\|_{H_{\infty,\beta}} &= \left\| \Omega_{\beta,1}\left(\mathbf{G}_c - \hat{\mathbf{G}}_2\right) \right\|_{h_{\infty,1}} \\ &= \left\| \Omega_{\beta,1}(\mathbf{G}_c) - \Omega_{\beta,1}\left(\hat{\mathbf{G}}_2\right) \right\|_{h_{\infty,1}} \\ &= \left\| \mathbf{G}_d - \hat{\mathbf{G}}_d \right\|_{h_\infty} \end{aligned}$$

Because $\mathbf{G}_d$ and $\hat{\mathbf{G}}_d$ are asymptotically stable and is the reduced system received $\hat{\mathbf{G}}_d$ by using the balanced truncation algorithm of $\mathbf{G}_d$. We have:

$$\left\| \mathbf{G}_d - \hat{\mathbf{G}}_d \right\|_{h_\infty} \le 2(\pi_{r+1} + ... + \pi_n),$$

where $\pi_{r+1} + ... + \pi_n$ are the Hankel singular values of $\mathbf{G}_d(z)$.

IV. CASE STUDY

Consider the 15-order unstable system described by the transfer function as follows:

$$\mathbf{G}_c(s) = \frac{\mathbf{A}(s)}{\mathbf{B}(s)}$$

where

$$\begin{aligned} \mathbf{A}(s) = &-s^{15} - 51.76s^{14} - 1239s^{13} - 1.82\times10^4 s^{12} \\ &-1.838\times10^5 s^{11} - 1.352\times10^6 s^{10} - 7.487\times10^6 s^9 \\ &-3.18\times10^7 s^8 - 1.044\times10^8 s^7 - 2.655\times10^8 s^6 \\ &-5.182\times10^8 s^5 - 7.631\times10^8 s^4 - 8.212\times10^8 s^3 \\ &-6.102\times10^8 s^2 - 2.802\times10^8 s - 6.004\times10^7 \end{aligned}$$

$$\begin{aligned} \mathbf{B}(s) = &\ 2.23\times10^{-7} s^{15} + 0.0004561s^{14} + 0.02061s^{13} \\ &+0.4153s^{12} + 4.912s^{11} + 37.92s^{10} + 200.9s^9 + 746.8s^8 \\ &+1948s^7 + 3488s^6 + 4064s^5 + 2715s^4 \\ &+693.2s^3 - 105.4s^2 + 7.276\times10^{-12}s \end{aligned}$$

Performing unstable continuous order reduction according to algorithm 1 in part II, we have the following results:

Performing order reduction for unstable continuous system $\mathbf{G}_c(s)$ according to Algorithm 1, presented in Section II, we obtain the following results:

TABLE I. ORDER REDUCTION RESULTS FOR UNSTABLE CONTINUOUS SYSTEM $\mathbf{G}_c(s)$ ACCORDING TO ALGORITHM 1

Order	β-stable continuous reduced-order system, $\mathbf{G}_{cr}(s)$	Error
5	$\dfrac{-4.485\times10^6 s^5 - 6.804\times10^7 s^4 - 4.123\times10^8 s^3 - 1.235\times10^9 s^2 - 1.816\times10^9 s - 1.09\times10^9}{s^5 + 2009s^4 + 1.833\times10^4 s^3 - 1913s^2 + 1.819\times10^{-9} s - 1.282\times10^{-10}}$	5.9734×10^{-7}
4	$\dfrac{-4.485\times10^6 s^4 - 2.658\times10^7 s^3 - 1.245\times10^8 s^2 - 1.754\times10^8 s - 1.23\times10^8}{s^4 + 2000s^3 + 188.4s^2 - 3.561s - 0.1971}$	2.2199×10^3
3	$\dfrac{-4.485\times10^6 s^3 - 2.627\times10^7 s^2 + 1.044\times10^8 s - 4.079\times10^8}{s^3 + 2000s^2 - 462s + 31.68}$	3.3272×10^5

For the sake of brevity and convenience, we make the following convention: a β-stable continuous r-th reduced-order system is called an r-th order system; an unstable continuous 15th-order system is called a 15th-order system.

From Table I, it can be seen that the 5th-order reduced system has a much smaller reduced-order error than the 4th-order reduced system (about 2.59×10^{-10} times) and the 3rd-order reduced system (1.795×10^{-13} times). Here, the transfer function of the reduced-order error of the r-th order system is determined by the difference in the transfer function of the 15th-order and r-th order system.

To compare and clarify the result of order reduction, we use the following graphs:

Comments:

From Fig. 3, we see that:

For $0 < t < 50$ s, the reduced-order error of the 5th, 4th, and 3rd-order reduced systems are all small, as shown in Fig. 1(a-d).

For $t > 50$s, the reduced-order error of the reduced-order systems starts to increase, in which the error rate of the 5th-order reduced system is the lowest, the error rate of the 3rd-order reduced system is the largest.

Comparing the magnitude of the reduced-order error, it is seen that the reduced-order error of the 5th-order reduced system is the smallest, the reduced-order error of the 3rd-order reduced system is the largest, as shown in

Fig. 3(d). This result is consistent with the results of the reduced-order error in Table I.

Fig. 3(d): From a time interval greater than 50s, compared to the 3rd-order reduced system, the 5th and 4th-order reduced systems have much higher reduced-order errors. Besides, the error characteristic of the 5th and 4th-reduced-order systems are almost identical.

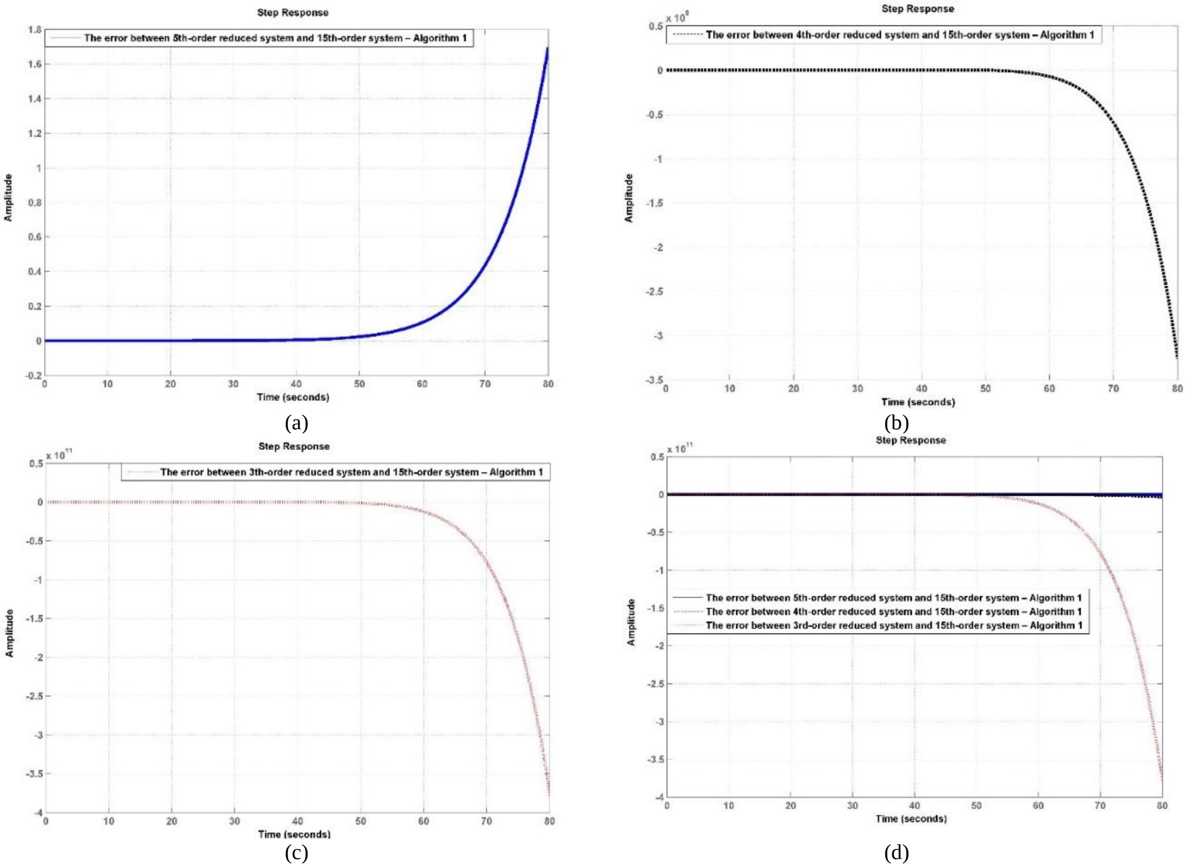


Figure 3. Rreduced-order systems according to Algorithm 1.

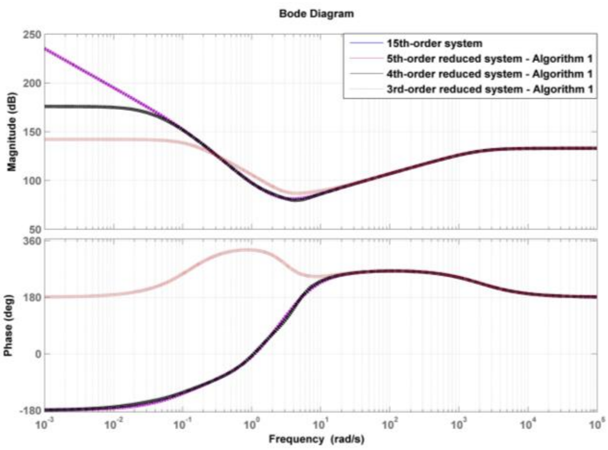


Figure 4. Bode diagrams of 15th-order and reduced-order systems according to Algorithm 1.

From Fig. 4, we see that:

The 5th-order reduced system has frequency phase and amplitude responses that almost coincide with those of the 15th-order system.

The 4th-order reduced system has a frequency phase response that almost coincides with the 15th-order system. The frequency phase response of the 4th-order reduced system only deviates from that of the original system for $\omega \leq 0.05$ rad/s. The smaller the frequency, the larger the deviation. For $\omega > 0.05$ rad/s, the frequency phase and amplitude responses of the 4th-order reduced system coincide with those of the 15th-order system.

For $\omega \leq 20$ rad/s, The frequency phase and amplitude responses of the 3rd-order reduced system are different from the 15th-order system. The smaller the frequency, the more significant the difference. For $\omega \leq 20$ rad/s, the 3rd-order reduced system has the frequency phase and amplitude responses matching the 5th-order reduction system.

Overall we see that the lower the order of the reduced-order system, the higher the reduced-order error and bode response error. From the above comments, we can evaluate the reduced-order systems according to Algorithm 1 as follows. To choose the best reduced-order system among three reduced-order systems according to Algorithm 1 to replace the 15th-order system, we can choose the 5th-order reduced system. The 4th-order reduced system can also be chosen to replace the 15th-order system if we accept a larger reduced-order error, but the bode response coincides with the 15th-order system. The 3rd-order reduced system should not be chosen to replace the 15th-order system because the reduced-order error and the deviation of the bode plot of the system compared with the 15th-order system are very large.

Performing order reduction of unstable system according to Algorithm 2 in Section III, we obtain the following results:

TABLE II. ORDER REDUCTION RESULTS FOR UNSTABLE CONTINUOUS SYSTEM $\mathbf{G}_c(s)$ ACCORDING TO ALGORITHM 2

Order	β-stable continuous reduced-order system, $\mathbf{G}_{cr}(s)$	Error
5	$\dfrac{-4.485\times10^{6}s^{5}-6.804\times10^{7}s^{4}-4.123\times10^{8}s^{3}-1.235\times10^{9}s^{2}-1.816\times10^{9}s-1.09\times10^{9}}{s^{5}+2009s^{4}+1.833\times10^{4}s^{3}-1913s^{2}+4.063\times10^{-11}s-4.696\times10^{-12}}$	1.1707×10^{-6}
4	$\dfrac{-4.485\times10^{6}s^{4}-3.063\times10^{7}s^{3}-1.158\times10^{8}s^{2}-1.824\times10^{8}s-1.186\times10^{8}}{s^{4}+2004s^{3}-204.6s^{2}-0.4293s+0.02271}$	2.0075×10^{3}
3	$\dfrac{-4.485\times10^{6}s^{3}-2.091\times10^{8}s^{2}+2.869\times10^{8}s-4.905\times10^{8}}{s^{3}+2140s^{2}-472s+30.59}$	2.3528×10^{5}

From Table II, it can be seen that the 5th-order reduced system has a much smaller reduced-order error than the 4th-order reduced system (about 0.583×10^{-9} times) and the 3rd-order reduced system (0.853×10^{-11} times). Here, the transfer function of the reduced-order error of an r-th order system is determined by the difference in the transfer function of the 15th-order and the r-th order system.

To compare and clarify the result of order reduction, we use the following graphs:

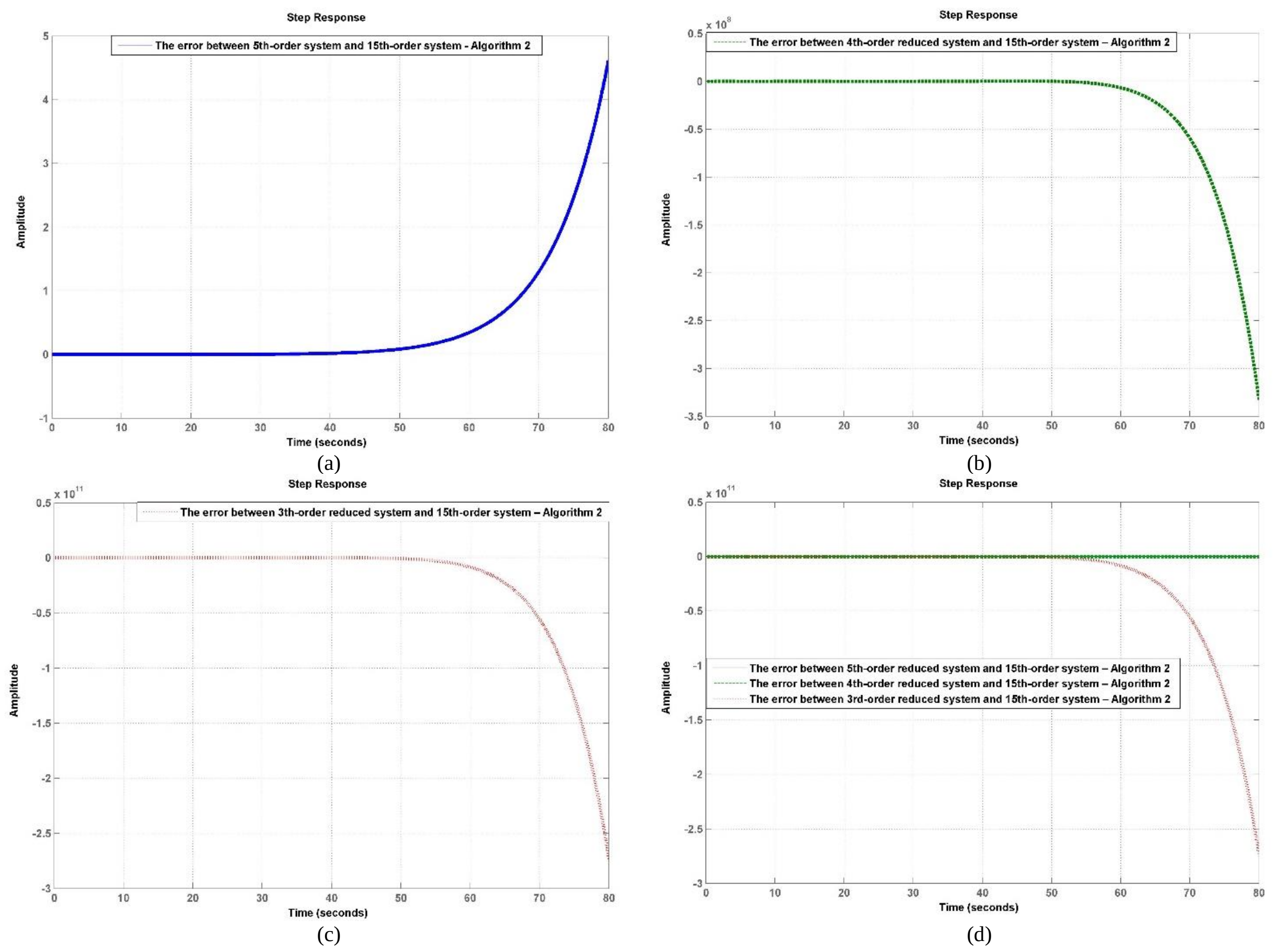


Figure 5. Reduced-order error of reduced-order systems according to Algorithm 2.

Comments:

From Fig. 5, we see that:

For $0 < t < 50$ s, the reduced-order error of the 5th, 4th, and 3rd-order reduced systems are all small, as shown in Fig. 5(a-d).

For $t > 50$ s, the reduced-order error of the reduced-order systems starts to increase, in which the error rate of the 5th-order reduced system is the lowest, the error rate of the 3rd-order reduced system is the largest.

Comparing the magnitude of the reduced-order error, it is seen that the reduced-order error of the 5th-order reduced system is the smallest, the reduced-order error of the 3rd-order reduced system is the largest, as shown in Fig. 5(d). This result is consistent with the results of the reduced-order error in Table II.

Fig. 5(d): From a time interval greater than 50 s, compared to the 3rd-order reduced system, the 5th and 4th-order reduced systems have much higher reduced-order errors. Besides, the error characteristic of the 5th and 4th-order reduced systems are almost identical.

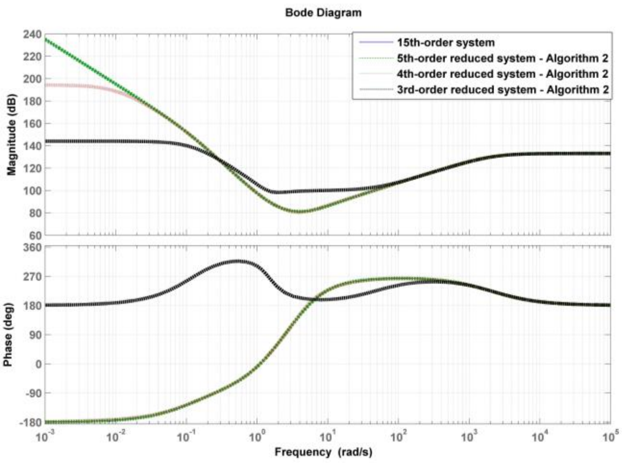


Figure 6. Bode diagrams of 15th order and reduced-order systems according to Algorithm 2.

From Fig. 6, we see that:

The 5th-order reduced system has frequency phase and amplitude responses that almost coincide with those of the 15th-order system.

The 4th-order reduced system has a frequency phase response that almost coincides with the 15th-order system. The frequency phase response of the 4th-order reduced system only deviates from that of the original system for $\omega \leq 0.21$ rad/s. The smaller the frequency, the larger the deviation. For $\omega > 0.21$ rad/s, the frequency phase and amplitude responses of the 4th-order reduced system coincide with those of the 15th-order system.

For $\omega \leq 0.491$ rad/s, the frequency phase responses of the 3rd-order reduced system are different from the 15th-order system. The smaller the frequency, the more significant the difference. For $\omega > 0.491$ rad/s, the 3rd-order reduced system has the frequency phase responses matching the 5th-order reduced system. For $\omega \leq 88.6$ rad/s, the frequency amplitude responses of the 3rd-order reduced system are different from the 15th-order system. The smaller the frequency, the more significant the difference. For $\omega > 88.6$ rad/s, the 3rd-order reduced system has the frequency phase responses matching the 5th-order reduced system.

Overall we see that the lower the order of the reduced-order system, the higher the reduced-order error and bode response error. From the above comments, we can evaluate the reduced-order systems according to Algorithm 2 as follows. To choose the best reduced-order system among three reduced-order systems according to Algorithm 2 to replace the 15th-order system, we can choose the 5th-order reduced system.

The 4th-order reduced system can also be chosen to replace the 15th-order system if we accept a larger reduced-order error, but the bode response coincides with the 15th-order system. The 3rd-order reduced system should not be chosen to replace the 15th-order system because the reduced-order error and the deviation of the bode plot of the system compared with the 15th-order system are very large.

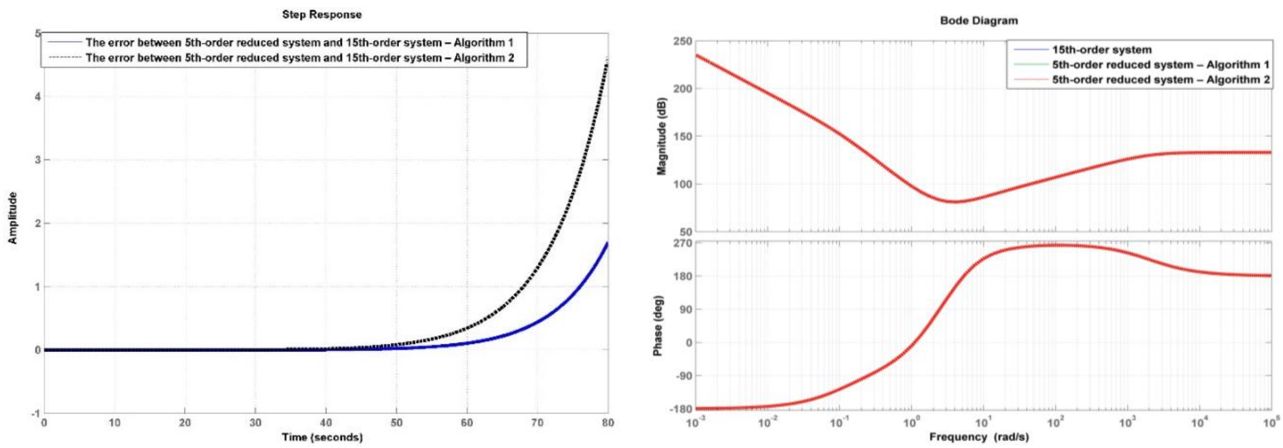
Comparing the results of order reduction in Table I and Table II, we see that:

The reduced-order error of the 5th-order reduced system according to Algorithm 1 is smaller than that of the 5th-order reduced system according to Algorithm 2 ($5.9734\times10^{-7} < 1.1707\times10^{-6}$).

+ The reduced-order error of the 4th-order reduced system according to Algorithm 1 is larger than that of the 4th-order reduced system according to Algorithm 2 ($2.2199\times10^{3} > 2.0075\times10^{3}$).

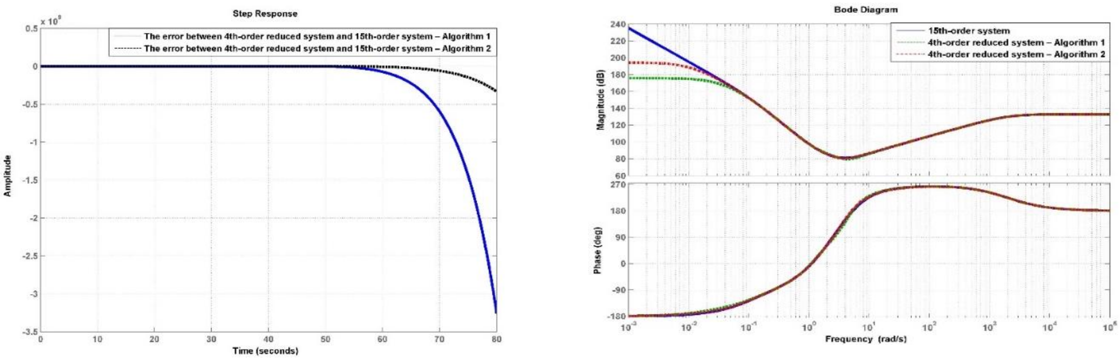
+ The reduced-order error of the 3rd-order reduced system according to Algorithm 1 is larger than that of the 4th-order reduced system according to Algorithm 2 ($3.3272\times10^{5} > 2.3528\times10^{5}$).

To clarify the comparison results between the reduced-order systems according to the two algorithms, we use error graph and bode graph as follows.



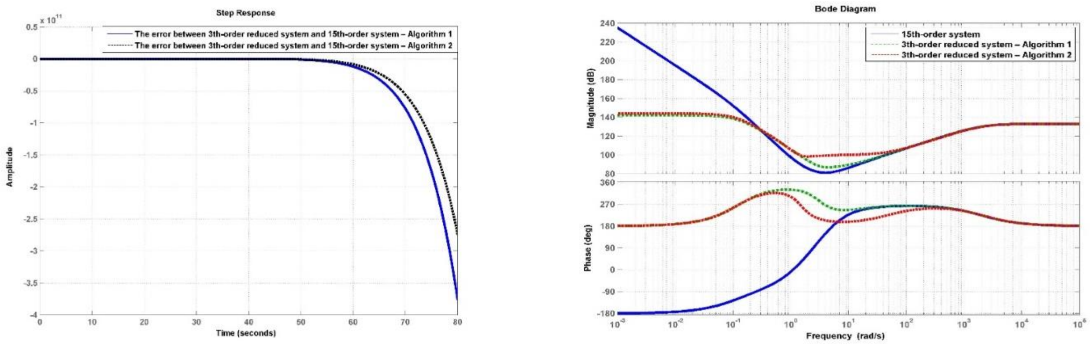
(a) (b)

Figure 7. Reduced-order error and bode plot of 5th-order reduced system according to 2 algorithms.



(a) (b)

Figure 8. Reduced-order error and bode plot of 4th-order reduced system according to 2 algorithms.



(c) (d)

Figure 9. Reduced-order error and bode plot of 3rd-order reduced system according to 2 algorithms.

From Figs. 7–9 and error evaluation in Table I and II, we have the following assessment:

Compare the 5th-order controller according to 2 algorithms, we should choose the 5th-order controller according to Algorithm 1 to replace the 15th-order controller.

Compare the 4th-order controller according to 2 algorithms, we should choose the 4th order controller according to Algorithm 4 to replace the 15th-order controller.

Compare the 3rd-order controller according to 2 algorithms, if the priority is to reduce the reduced-order error, we should choose the 3rd-order controller according to Algorithm 2 to replace the 15th-order controller; if the bode plot is preferred, we should select the 3rd-order controller according to Algorithm 1 to replace the 15th-order controller.

Thus, considering the three reduced-order systems, Algorithm 1 and Algorithm 2 have similar reduced-order efficiency.

V. Conclusion

The paper presents two unstable linear order reduction algorithms based on the mapping. Algorithm 1 is built on the basis of continuous-continuous mapping, and Algorithm 2 is built on the basis of continuous-discrete mapping. The common basis of the two algorithms seeks to exploit the mapping to convert from an unstable system to a stable system so that a balanced truncation algorithm can be applied. Applying two algorithms to reduce the order of an unstable 15th-order system shows that: the 5th-order reduced system has a smaller reduced-order error and a smaller bode plot deviation than those of the 4th-order reduced and the 3rd-order reduced system. Considering the three reduced-order systems, the order reduction efficiency of the two algorithms is almost equivalent. The results in the illustrative example have shown the correctness of algorithms and opened usability in practice.

Data Availability

This publication is supported by multiple datasets, which are available at locations cited in the reference section.

Conflict of Interest

The authors declare that they have no conflicts of interest.

Author Contributions

Vu Ngoc Kien devised the project, the main conceptual ideas and verified the simulation data. Nguyen Hong Quang contributed to analyzing the results and writing the manuscript, data curation, investigation editing, and supervision. All authors have read and agreed to the published version of the manuscript.

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