Solution of Fuzzy Volterra Integral Equations In A Bernstein Polynomial Basis

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***Abstract*— In this paper, we have used the parametric form of fuzzy number and convert a fuzzy Volterra integral equation to a system of integral equations in crisp case. We present a numerical method for solving fuzzy Volterra integral equations of the second kind. The proposed method is based on approximating unknown function with Bernstein's approximation. This method using simple computation with quite acceptable approximate solution. However, accuracy and efficiency are dependent on the size of the set of Bernstein polynomials. Furthermore we get an estimation of error bound for this method.**

***Index Terms*— Fuzzy integral equations, System of Volterra integral equation, Bernstein polynomial**

I. Introduction

The solutions of integral equations have a major role in the field of science and engineering. A physical even can be modelled by the differential equation, an integral equation. Since few of these equations cannot be solved explicitly, it is often necessary to resort to numerical techniques which are appropriate combinations of numerical integration and interpolation [1, 2]. There are several numerical methods for solving linear Volterra integral equation [3]. Kauthen in [4] used a collocation method to solve the Volterra- Fredholm integral equation numerically. Maleknejad and et al. in [5] obtained a numerical solution of Volterra integral equations by using Bernstein Polynomials.

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [6], Dubois and Prade [7]. We refer the reader to [8] for more information on fuzzy numbers and fuzzy arithmetic. The topics of fuzzy integral equations (FIE) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recent years. The fuzzy mapping function was introduced by Chang and Zadeh [9]. Later, Dubois and Prade [10] presented an elementary fuzzy calculus based on the extension principle also the concept of integration of fuzzy functions was first introduced by Dubois and Prade [10]. Babolian et al., Abbasbandy et al. in [11, 12] obtained a numerical solution of linear Fredholm fuzzy integral equations of the second kind. Also, the fuzzy integral equations have been studied by several authors [13, 14, 15].

In this paper, we present a novel and very simple numerical method based upon Bernstein's approximation for solving Volterra fuzzy integral equations.

II. Preliminaries

In this section the basic notations used in fuzzy calculus and Bernstein polynomials are introduced. We start by defining the fuzzy number.

Definition 1. [16] A fuzzy number is a fuzzy set $u:\mathbb{R}^1 \to I=[0,1]$ such that

i. u is upper semi-continuous;

ii. $u(x)=0$ outside some interval $[a,d]$;

iii. There are real numbers b and c, $a \le b \le c \le d$, for which

1. $u(x)$ is monotonically increasing on $[a,b]$,
2. $u(x)$ is monotonically decreasing on $[c,d]$,
3. $u(x)=1, b \le x \le c$.

The set of all the fuzzy numbers (as given in definition 1) is denoted by E^1.

An alternative definition which yields the same E^1 is given by Kaleva [17, 18].

Definition 2. A fuzzy number u is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

i. $\underline{u}(r)$ is a bounded monotonically increasing, left continuous function on $(0,1]$ and right continuous at 0;

ii. $\overline{u}(r)$ is a bounded monotonically decreasing, left continuous function on $(0,1]$ and right continuous at 0;

iii. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

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A crisp number r is simply represented by $\underline{u}(\alpha) = \overline{u}(\alpha) = r, 0 \le \alpha \le 1.$ The set of all the fuzzy numbers is denoted by E^1.

For arbitrary $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathsf{R}$ we define addition and multiplication by k as

$$\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r)),$$
$$\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r)),$$
$$\underline{ku}(r) = k\underline{u}(r), \overline{ku}(r) = k\overline{u}(r), \text{ if } k \ge 0,$$
$$\underline{ku}(r) = k\overline{u}(r), \overline{ku}(r) = k\underline{u}(r), \text{ if } k < 0.$$

Remark 1. [12] Let $u = (\underline{u}(r), \overline{u}(r)), 0 \le r \le 1$ be a fuzzy number, we take

$$u^c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2},$$
$$u^d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}.$$

It is clear that $u^d(r) \ge 0$, $\underline{u}(r) = u^c(r) - u^d(r)$ and $\overline{u}(r) = u^c(r) + u^d(r)$, also a fuzzy number $u \in E^1$ is said symmetric if $u^c(r)$ is independent of r for all $0 \le r \le 1.$

Remark 2. Let $u = (\underline{u}(r), \overline{u}(r)),\ v = (\underline{v}(r), \overline{v}(r))$ and also k, s are arbitrary real numbers. If $w = ku + sv$ then

$$w^c(r) = ku^c(r) + sv^c(r),$$
$$w^d(r) = |k| u^d(r) + |s| v^d(r).$$

Definition 3. [19] For arbitrary fuzzy numbers $u, v,$ we use the distance

$$D(u,v) = sup_{0 \le r \le 1} max\{|\overline{u}(r) - \overline{v}(r)|, |\underline{u}(r) - \underline{v}(r)|\}$$

and it is shown that (E^1, D) is a complete metric space [20].

The Bernstein's approximation, $B_n(f)$ to a real function $f : [0,1] \to \mathsf{R}$ is the polynomial

$$B_n(f(x)) = \sum_{i=0}^{n} f(\frac{i}{n}) P_{n,i}(x), \quad (1)$$

where

$$P_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \ i = 0,1,...,$$

There are $n+1$ n th-degree polynomials. For convenience, we set $P_{n,i}(x) = 0,$ if $i < 0$ or $i > n.$ It can be readily shown that each of the Bernstein polynomials is positive.

Theorem 1. For all functions f in $C[0,1]$, the sequence $B_n(f); n = 1,2,3,...$ converges uniformly to f, where B_n is defined by Eq. (1).

Proof. See [21].

This theorem follows that, for any $f \in C[0,1]$ and for any $\varepsilon > 0$, there exists n such that the inequality $\|B_n(f) - f\| < \varepsilon,$ holds.

We suppose $\| . \|$ be the max norm on $[0,1]$, then the error bound

$$| B_n(f(x)) - f(x) | \le \frac{1}{2n} x(1-x) \| f'' \|, \quad (2)$$

given in [22], shows that the rate of convergence is at least $\frac{1}{n}$ for $f \in C[0,1]$. On the other hand, the asymptotic formula

$$lim_{n \to \infty} n(B_n(f(x)) - f(x)) = \frac{1}{2} x(1-x) f''(x), \quad (3)$$

due to Voronovskaya [23] shows that for $x \in (0,1)$ with $f' \ne 0,$ the rate of convergence is precisely $\frac{1}{n}.$

III. Fuzzy Volterra Integral equation

The Fuzzy Volterra integral equations of the second kind (FVIE-2) is [24]

$$F(x) = G(x) + \lambda \int_0^x k(x,t) F(t) dt, \quad 0 \le x \le 1. \quad (4)$$

where $\lambda > 0, k(x,t)$ is a kernel function and $G(x)$ is a fuzzy function. If $G(x)$ is a fuzzy function these equation may only possess fuzzy solution. Sufficient conditions for the existence of a unique solution to the fuzzy Volterra integral equation are given in [24].

Now, we introduce parametric form of a FVIE-2 with respect to Definition 2. Let $(\underline{G}(x;r), \overline{G}(x;r))$ and $(\underline{F}(x;r), \overline{F}(x;r)), 0 \le r \le 1$ are parametric form of $G(x)$ and $F(x)$, respectively then, parametric form of FVIE-2 is as follows:

$$\underline{F}(x;r) = \underline{G}(x;r) + \lambda \int_0^x \underline{k(x,t) F(t;r)} dt, \quad 0 \le x \le 1, \quad (5)$$

$$\overline{F}(x;r) = \overline{G}(x;r) + \lambda \int_0^x \overline{k(x,t) F(t;r)} dt. \quad (6)$$

Suppose $k(x,t)$ be continuous and for fix t, $k(x,t)$ changes its sign in finite points as x_j where $x_j \in [0, x].$ For example, let $k(x,t)$ be nonnegative over $[0, x_1]$ and negative over $[x_1, x]$, therefore from Eqs. (5) and (6), we have

$$\underline{F}(x;r) = \underline{G}(x;r) + \lambda\int_0^{x_1} k(x,t)\underline{F(t;r)}dt$$

$$+ \lambda\int_{x_1}^{x} k(x,t)\overline{F(t;r)}dt, 0 \le x \le 1,\ 0 \le r \le 1,$$

$$\overline{F}(x;r) = \overline{G}(x;r) + \lambda\int_0^{x_1} k(x,t)\overline{F(t;r)}dt$$

$$+ \lambda\int_{x_1}^{x} k(x,t)\underline{F(t;r)}dt, 0 \le x \le 1,\ 0 \le r \le 1.$$

By referring to Remark 2 we have

$$F^c(x;r) = G^c(x;r) + \lambda\int_0^x k(x,t)F^c(t;r)dt, \quad 0 \le x \le 1, \tag{7}$$

$$F^d(x;r) = G^d(x;r) + \lambda\int_0^x |k(x,t)|F^d(t;r)dt,\ 0 \le x \le 1. \tag{8}$$

It is clear that we must solve two crisp Volterra integral equation of the second kind provided that each of Eqs. (7) and (8) have solution.

We consider the Volterra integral equations of the second kind given by,

$$F^c(x;r) = G^c(x;r) + \lambda\int_0^x k(x,t)F^c(t;r)dt,$$

$$F^d(x;r) = G^d(x;r) + \lambda\int_0^x |k(x,t)|F^d(t;r)dt,$$

$$0 \le x \le 1,$$

where $F^c(x;r)$ and $F^d(x;r)$ are the unknown crisp function to be determined, $k(x,t)$ is a continuous function on the square $[0,1]^2$ and integrable function, $G^c(x;r)$ and $G^d(x;r)$ being the known crisp functions.

To determine an approximate the unknown function of Eq. (4), we approximate with Bernstein's approximation

$$B_n(F(x)) = G(x) + \lambda\int_0^x k(x,t)B_n(F(t))dt, \tag{9}$$

therefore, we approximate the unknown functions $F^c(x;r)$ and $F^d(x;r)$ by

$$B_n(F^c(x;r)) = \sum_{i=0}^{n} F^c(\frac{i}{n};r)P_{n,i}(x), \tag{10}$$

and

$$B_n(F^d(x;r)) = \sum_{i=0}^{n} F^d(\frac{i}{n};r)P_{n,i}(x), \tag{11}$$

where

$$P_{n,i}(x) = \binom{n}{i}x^i(1-x)^{n-i},\ i = 0,1,...,$$

Let $(\underline{B_n(F(x;r))}, \overline{B_n(F(x;r))})$, $0 \le r \le 1$ is a parametric form of $B_n(F(x))$, then we have:

$$\underline{B_n(F(x;r))} = \sum_{i=0}^{n} \underline{F(\frac{i}{n};r)}P_{n,i}(x), 0 \le r \le 1,$$

$$\overline{B_n(F(x;r))} = \sum_{i=0}^{n} \overline{F(\frac{i}{n};r)}P_{n,i}(x), 0 \le r \le 1$$

By referring to Remark 2, we have the following equations

$$\sum_{i=0}^{n} F^c(\frac{i}{n};r)\binom{n}{i}(x^i(1-x)^{n-i}$$

$$-\lambda\int_0^x k(x,t)t^i(1-t)^{n-i}dt) = G^c(x;r),$$

$$0 \le r \le 1, \tag{12}$$

$$\sum_{i=0}^{n} F^d(\frac{i}{n};r)\binom{n}{i}((x^i(1-x)^{n-i}$$

$$-\lambda\int_0^x |k(x,t)|t^i(1-t)^{n-i}d, \tag{13}$$

In order to find $F^c(\frac{i}{n};r)$ and $F^d(\frac{i}{n};r)$ for $i = 0,1,...,n$, we now put $x = x_j$, $j = 0,1,...,n$ in (12) and (13), x_j's being chosen as suitable distinct points in $(0,]$, and x_0 is taken near 0 such that $0 < x_0 < x_1 < ... < x_n = 1$. Putting $x = x_j$ we obtain in short form two linear systems

$$A_1X_1 = Y_1, \tag{14}$$

where

$$A_1 = [\binom{n}{i}(x_j^i(1-x_j)^{n-i}$$

$$-\lambda\int_0^{x_j} k(x_j,t)t^i(1-t)^{n-i}dt)],$$

$$i, j = 0,1,...,n,$$

$$X_1 = [F^c(\frac{i}{n};r)]^t,\ 0 \le r \le 1,\ i = 0,1,...,n,$$

$$Y_1 = [G^c(x_j;r)]^t,\ 0 \le r \le 1,\ j = 0,1,...,n,$$

and also

$$A_2X_2 = Y_2, \tag{15}$$

where

$$A_2 = [\binom{n}{i}(x_j^i(1-x_j)^{n-i} - \lambda\int_0^{x_j} |k(x_j,t)|\, t^i(1-t)^{n-i}dt)],$$
$$i, j = 0,1,...,n,$$

$$X_1 = [F^d(\frac{i}{n};r)]^t,\ 0 \le r \le 1,\ i = 0,1,...,n,$$

$$Y_1 = [G^d(x_j;r)]^t,\ 0 \le r \le 1,\ j = 0,1,...,n.$$

In general we cannot be able to carry out analytically the integrations, involved. We compute the integral that exist in A_1's formula and A_2's formula numerically. Now we can show $F^c(\frac{i}{n};r)$ and $F^d(\frac{i}{n};r)$ by $F_n^c(\frac{i}{n};r), i = 0,1,...,n$ and $F_n^d(\frac{i}{n};r), i = 0,1,...,n,$ respectively that are our solutions in nodes $x_j, j = 0,1,...,n$ and by substituting them in Eqs. (10) and (11) we can find $B_n(F_n^c(x_j;r))$ and $B_n(F_n^d(x_j;r))\ j = 0,1,...,n$ that are solution for integral equations (7) and (8).

We give error bound for this solution in the following theorem.

Theorem 2. Consider the crisp Volterra integral equations of the second kind (7) and (8). Assume that $k(x,t)$ is continuous on the square $[0,1]^2$ and the solution of the equations belong to $(C^\alpha \cap L^2)([0,1])$ for some $\alpha > 2$. If A_1 and A_2 invertible then

$$sup_{x_i\in[0,1]} D(F(x_i), B_n(F_n(x_i))) \le$$

$$sup_{r\in[0,1]}[\frac{1}{8n}((1+s)\|AA_1^{-1}\|\|\ F'''^c(r)\ \| + \|\ F_n'''^c(r)\ \|)$$

$$\frac{1}{8n}((1+s)\|\ A_2^{-1}\ \|\|\ F'''^d(r)\ \| + \|\ F_n'''^d(r)\ \|)],$$

where $x_i = \frac{i}{n}, i = 0,1,...,n,\ F(x)$ is exact solution of FVIE-2 and $s = sup_{x,t\in[0,1]} |\lambda k(x,t)|$.

Proof. We have

$$sup_{x_i\in[0,1]} |F^\dagger(x_i;r) - B_n(F_n^\dagger(x_i;r))| =$$
$$sup_{x_i\in[0,1]} |F^\dagger(x_i;r) - F_n^\dagger(x_i;r) + F_n^\dagger(x_i;r) -$$
$$B_n(F_n^\dagger(x_i;r))| \le$$
$$sup_{x_i\in[0,1]} |F^\dagger(x_i;r) - F_n^\dagger(x_i;r)| + sup_{x_i\in[0,1]}$$
$$|F_n^\dagger(x_i;r) - B_n(F_n^\dagger(x_i;r))|,$$

where $\dagger$ means we have this equation for c and d together, independently. From relation (2) we have the following bound

$$sup_{\ x\in[0,1]}\ |\ F_n^\dagger(x;r) - B_n(F_n^\dagger(x;r))\ | \le$$
$$\frac{1}{2n}x(1-x)\|\ F_n^\dagger\ \| \le \frac{1}{8n}\|\ F_n^\dagger\ \|, \quad (17)$$

then it is enough to find a bound for $sup_{x_i\in[0,1]} |F^\dagger(x_i;r) - F_n^\dagger(x_i;r)|$. For numerically solving integral equations (7) and (8) by using Bernstein's approximation, because from Theorem 1 we know that for any $F^\dagger \in C[0,1]$ and for any $\varepsilon > 0$, there exists n such that the inequality $\|\ B_n(F^\dagger) - F^\dagger\ \| < \varepsilon,$ holds so we can write integral equations (7) and (8) as

$$G^c(x;r) = B_n(F^c(x;r)) - \lambda\int_0^x k(x,t)B_n(F^c(t;r))dt,$$

and

$$G^d(x;r) = B_n(F^d(x;r)) - \lambda\int_0^x |k(x,t)|\, B_n(F^d(t;r))dt.$$

If we substitute $F_n^\dagger(x;r)$ instead of $F^\dagger(x;r)$ in above equation then the right-hand side of integral equation is exchanged by a new function that we denote it by $\hat{G}^\dagger(x;r)$. So we have,

$$\hat{G}^c(x;r) = B_n(F_n^c(x;r)) - \lambda\int_0^x k(x,t)B_n(F_n^c(t;r))dt,$$

and

$$\hat{G}^d(x;r) = B_n(F_n^d(x;r)) - \lambda\int_0^x |k(x,t)|\, B_n(F_n^d(t;r))dt,$$

Consequently we have

$$sup_{x_i\in[0,1]} |F^c(x_i;r) - F_n^c(x_i;r)| \le \|\ A_1^{-1}\ \|$$
$$max\ |G^c(x_i;r) - \hat{G}^c(x_i;r)|,$$
$$i = 0,1,...,n, \quad (18)$$

$$sup_{\ x_i\in[0,1]}\ |F^c(x_i;r) - F_n^c(x_i;r)| \le \|\ A_1^{-1}\ \|$$
$$max\ |G^c(x_i;r) - \hat{G}^c(x_i;r)|,$$
$$i = 0,1,...,n,$$

and

$$sup_{\ x_i\in[0,1]}\ |F^d(x_i;r) - F_n^d(x_i;r)| \le \|\ A_2^{-1}\ \|$$
$$max\ |G^d(x_i;r) - \hat{G}^d(x_i;r)|,$$
$$i = 0,1,...,n, \quad (19)$$

where $x_i = \frac{i}{n}, i = 0,1,...,n.$ For finding a bound for $max|G^{\dagger}(x_i;r) - \hat{G}^{\dagger}(x_i;r)|,$

we let

$$G^c(x;r) = F^c(x;r)) - \lambda\int_0^x k(x,t)F^c(t;r))dt,$$

$$G^d(x;r) = F^d(x;r)) - \lambda\int_0^x |k(x,t)|F^d(t;r))dt,$$

and

$$\hat{G}^c(x;r) = B_n(F^c(x;r)) - \lambda\int_0^x k(x,t)B_n(F^c(t;r))dt,$$
$$\hat{G}^d(x;r) = B_n(F^d(x;r)) - \lambda\int_0^x |k(x,t)|B_n(F^d(t;r))dt.$$

So that

$$G^c(x;r) - \hat{G}^c(x;r) = F^c(x;r) - B_n(F^c(x;r)) - \lambda\int_0^x k(x,t)(F^c(t;r) - B_n(F^c(t;r)))dt,$$

and

$$G^d(x;r) - \hat{G}^d(x;r) = F^d(x;r) - B_n(F^d(x;r)) - \lambda\int_0^x |k(x,t)|(F^d(t;r) - B_n(F^d(t;r)))dt,$$

if we let $sup_{x,t\in[0,1]}|\lambda k(x,t)| = s$, then we have

$$sup|G^c(x;r) - \hat{G}^c(x;r)| = sup|F^c(x;r) - B_n(F^c(x;r)) - \lambda\int_0^x k(x,t)(F^c(t;r) - B_n(F^c(t;r)))dt| \le$$
$$sup|F^c(x;r) - B_n(F^c(x;r))| + sup|\lambda\int_0^x k(x,t)(F^c(t;r) - B_n(F^c(t;r)))dt| \le$$
$$sup|F^c(x;r) - B_n(F^c(x;r))| + sup|\lambda k(x,t)(F^c(t;r) - B_n(F^c(t;r)))| \le$$
$$\frac{1}{2n}x(1-x)\|F''^c\| + s\frac{1}{2n}t(1-t)\|F''^c\| \le \frac{1}{8n}(1+s)\|F''^c\|,$$

and

$$sup|G^d(x;r) - \hat{G}^d(x;r)| \le \frac{1}{8n}(1+s)\|F''^d\|,$$

so by substituting this bound in the inequality (18) and (19) we have,

$$sup_{x_i\in[0,1]}|F^c(x_i;r) - F_n^c(x_i;r)| \le \frac{1}{8n}(1+s)\|A_1^{-1}\|\|F''\|, \quad (20)$$

then from relations (16), (17),(20) and (21) we have

$$sup_{x_i\in[0,1]}|F^c(x_i;r) - B_n(F_n^c(x_i;r))| \le \frac{1}{8n}((1+s)\|A_1^{-1}\|\|F''^c\| + \|F_n''^c\|), \quad (22)$$

and

$$sup_{x_i\in[0,1]}|F^d(x_i;r) - F_n^d(x_i;r)| \le \frac{1}{8n}(1+s)\|A_2^{-1}\|\|F''^d\|,$$
$$sup_{x_i\in[0,1]}|F^d(x_i;r) - B_n(F_n^d(x_i;r))| \le \frac{1}{8n}((1+s)\|A_2^{-1}\|\|F''^d\| + \|F_n''^d\|), \quad (23)$$

therefore by (22), (23) and Remark 1 we have

$$sup_{x_i\in[0,1]}|\underline{F(x_i;r)} - \underline{B_n(F_n(x_i;r))}| \le \frac{1}{8n}((1+s)\|A_1^{-1}\|\|F''^c\| + \|F_n''^c\|) + \frac{1}{8n}((1+s)\|A_2^{-1}\|\|F''^d\| + \|F_n''^d\|),$$

$$sup_{x_i\in[0,1]}|\overline{F(x_i;r)} - \overline{B_n(F_n(x_i;r))}| \le \frac{1}{8n}((1+s)\|A_1^{-1}\|\|F''^c\| + \|F_n''^c\|) + \frac{1}{8n}((1+s)\|A_2^{-1}\|\|F''^d\| + \|F_n''^d\|),$$

hence for all $r \in [0,1]$

$$max\{sup_{x_i\in[0,1]}[|\underline{F(x_i;r)} - \underline{B_n(F_n(x_i;r))}|, |\overline{F(x_i;r)} - \overline{B_n(F_n(x_i;r))}|]\}$$
$$\le \frac{1}{8n}((1+s)\|A_1^{-1}\|\|F''^c\| + \|F_n''^c\|) + \frac{1}{8n}((1+s)\|A_2^{-1}\|\|F''^d\| + \|F_n''^d\|),$$

and then

and the proof is completed.

Theorem 3. (Minc [25]) The inverse of a nonnegative matrix A_2 is nonnegative if and only if A_2 is a generalized permutation matrix.

$$sup_{x_i \in [0,1]} D(F(x_i), B_n(F_n(x_i))) \leq$$

$$sup_{r \in [0,1]} [\frac{1}{8n}((1+s)\| A_1^{-1} \|\| F''^c(r) \| + \| F_n''^c(r) \|) +$$

$$\frac{1}{8n}((1+s)\| A_2^{-1} \|\| F''^d(r) \| + \| F_n''^d(r) \|)],$$

Theorem 4. Let $(G(x_j)),\ j = 0,1,...,n$ is a fuzzy

$$x_j^i(1-x_j)^{n-i} \geq \lambda \int_0^{x_j} |k(x_j,t)| t^i (1-t)^{n-i} dt, i, j = 0,1,...n,$$

arbitrary vector, and A_2 is a generalized permutation matrix, then Eq. (9) has a fuzzy Bernstein approximation.

$$x_j^i(1-x_j)^{n-i} \geq \lambda \int_0^{x_j} |k(x_j,t)| t^i (1-t)^{n-i} dt, i, j = 0,1,...n,$$

Proof. By we have $A_2 \geq 0$. By Theorem 3 and our hypotheses, proof is completed.

A. Comparison with Other Methods

In this subsection, the shortcomings of the existing methods [5,12,26] for solving fuzzy integral equations are pointed out.

Maleknejad and et al. in [5] obtained a numerical solution of Volterra integral equations of the form

$$F(x) = G(x) + \lambda \int_0^x k(x,t)F(t)dt, \quad 0 \leq x \leq 1$$

where $\lambda > 0, k(x,t)$ and $G(x)$ are real function, by using Bernstein Polynomials. But in this paper we consider more general form of fuzzy Volterra integral equation (4) where $\lambda > 0, k(x,t)$ is a kernel function and $G(x)$ is a fuzzy function.

Abbasbandy and et al. in [12] considered linear Fredholm fuzzy integral equations of the second kind of the form

$$F(x) = G(x) + \lambda \int_0^a k(x,t)F(t)dt, \quad 0 \leq x \leq 1$$

where $\lambda > 0, k(x,t)$ is a kernel function and $G(x)$ is a fuzzy function. In this paper, if we consider $x = a$ in equation (4) then this paper transfer to [12].

Abbasbandy and et al. in [26] used the homotopy analysis method (HAM) to obtain solution of fuzzy integro-differential equation

$$F'(x) + F(x) + \lambda \int_0^x k(x,t)F(t)dt = G(x),$$

$$0 \leq x \leq 1.$$

But, in this paper we used Bernstein Polynomials to obtain solution of equation (4).

IV. NUMERICAL EXAMPLES

To illustrate the technique proposed in this paper, consider the following examples.

Example 4.1. We consider the fuzzy Volterra integral equation of the second kind given by,

$$F(x) = G(x) + \lambda \int_0^x F(t)dt, \quad 0 \leq x \leq 1,$$

where $\lambda = 1$

$$G(x) = (\underline{G(x;r)}, \overline{G(x;r)}) = (r, 2-r), 0 \leq r \leq 1.$$

The exact solution in this case is given by

$$F(x) = (\underline{F(x;r)}, \overline{F(x;r)}) = (re^x, (2-r)e^x), 0 \leq r \leq 1.$$

We can see that

$$G^c(x;r) = 1,\ G^d(x;r) = 1-r,\ 0 \leq r \leq 1.$$

According to Eqs. (7) and (8) we have the following two crisp Volterra integral equations

$$F^c(x;r) = 1 + \int_0^x F^c(t;r)dt,\ 0 \leq x \leq 1,\ 0 \leq r \leq 1,$$

$$F^d(x;r) = 1 - r + \int_0^x F^d(t;r)dt,\ 0 \leq x \leq 1, 0 \leq r \leq 1.$$

Now we approximate the unknown functions $F^c(x;r)$ and $F^d(x;r)$ by $B_n(F^c(x;r))$ and $B_n(F^d(x;r))$

TABLE I.
COMPUTED ERROR FOR EXAMPLES

n	Example 5.1	Example 5.2
	0.3513	0.5624
	0.0059	0.0015
	0.2844E-3	0.5848E-3

for $n = 1,2,3$.

We choose $x_0 = 10^{-10}$ and $x_j = \frac{j}{n}$ for $j = 1,..., n, n = 1,2,3.$ For this example, we use

$r = 0,0.1,\ldots,1,$ where we calculate the error of the exact solution and obtained solution of fuzzy Volterra integral equation with Bernstein approximation. Table 1 show the convergence behavior for $n = 1,2,3$. The exact and obtained solution of fuzzy Volterra integral equation

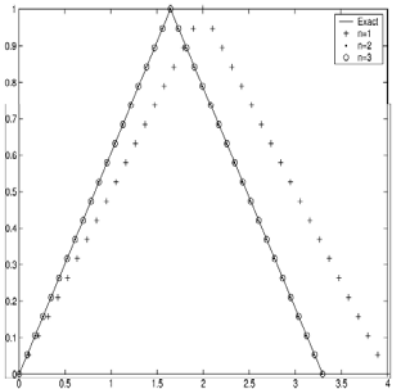

Figure 1. Compares the exact solution and obtained solutions

in this example at $x = 0.5$ for $n = 1,2,3$, are shown in Figure 1.

Example 4.2. We consider the fuzzy Volterra integral equation of the second kind given by,

$$F(x) = G(x) + \lambda\int_0^x k(x,t)F(t)dt, \quad 0 \le x \le 1$$

where $\lambda = 1$, $k(x,t) = sinhx$ and

$$G(x) = (\underline{G(x;r)}, \overline{G(x;r)}) =$$

$$((coshx + 1 - cosh^2 x)(r^2 + r),$$

$$(coshx + 1 - cosh^2 x)(4 - r^3 - r)), 0 \le r \le 1.$$

The exact solution in this case is given by

$$F(x) = (\underline{F(x;r)}, \overline{F(x;r)}) =$$

$$(coshx(r^2 + r), coshx(4 - r^3 - r)), 0 \le r \le 1.$$

We can see that

$$G^c(x;r) = \frac{(coshx + 1 - cosh^2 x)(4 + r^2 - r^3)}{2},$$

$$G^d(x;r) = \frac{(coshx + 1 - cosh^2 x)(4 - r^3 - r^2 - 2r)}{2},$$

$$0 \le r \le 1.$$

According to Eqs. (7) and (8) we have the following two crisp Volterra integral equations

$$F^c(x;r) = \frac{(coshx + 1 - cosh^2 x)(4 + r^2 - r^3)}{2} +$$

$$\int_0^x sinh(x)F^c(t;r)dt,$$

$$0 \le x \le 1, \ 0 \le r \le 1,$$

$$F^d(x;r) = \frac{(coshx + 1 - cosh^2 x)(4 - r^3 - r^2 - 2r)}{2}$$

$$+ \int_0^x sinh(x)F^d(t;r)dt,$$

$$0 \le x \le 1, \ 0 \le r \le 1.$$

Now we approximate the unknown functions $F^c(x;r)$ and $F^d(x;r)$ by $B_n(F^c(x;r))$ and $B_n(F^d(x;r))$ for $n = 1,2,3$.

We choose $x_0 = 10^{-10}$ and $x_j = \frac{j}{n}$ for $j = 1,...,n,\ n = 1,2,3.$ For this example, we use $r = 0,0.1,\ldots,1,$ where we calculate the error of the exact solution and obtained solution of fuzzy Volterra integral equation with Bernstein approximation. Table 1 show the convergence behavior for $n = 1,2,3$. The exact and obtained solution of fuzzy Volterra integral equation in this example at $x = 0.5$ for $n = 1,2,3$, are shown in Figure 2.

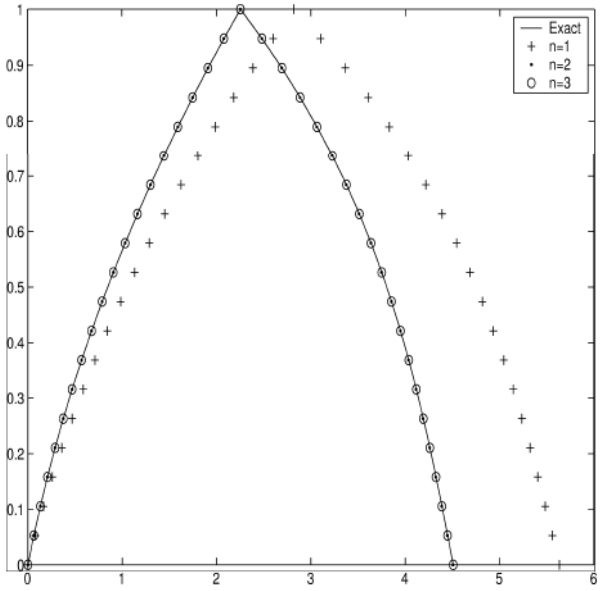

Figure 2. Compares the exact solution and obtained solutions

V. CONCLUSIONS

Here a very simple and straight method, based on approximation of the fuzzy unknown function of an fuzzy Volterra integral equation on the Bernstein polynomial basis is used. Our achieve results in this paper, show that Bernstein's approximation method for solving fuzzy Volterra integral equations of second kind, is very effective and the answers are trusty and their accuracy are high and we can execute this method in a computer simply.

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