

The Influence of J_2 on Formation Flying of Micro-Satellites in Low Near Equatorial Orbits

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Abstract—Taking into considerations the effect of the Earth's oblateness parameters, particularly J_2 , the present work assess such influence on the formation flight of micro satellites in near equatorial low orbits. The modified Clohessy-Wiltshire equations are reviewed to arrive at a convenient formulation as a set of linearized differential equations of motion to include the J_2 effects in the LVLH frame of reference. Comparison made on the orbit of twin-satellite formation flying with respect to that predicted by baseline Hill-Clohessy-Wiltshire equation, and to similar results in the literature, exhibit the plausibility of the work.

Index Terms—Orbital Mechanics, Gravitational Potential, Near-Equatorial Low Orbits, Spacecraft Formation Flying.

I. INTRODUCTION

Interest in the relative motion dynamics and control of spacecrafts in formation has grown *due to the need for deploying multiple spacecrafts flying in precise formations for Earth or space observation, orspace communications, and the affordability of smaller satellites, with capabilities equivalent or better than a single larger satellite, due to their modularity, simplicity, ease of launch and graceful degradation* (Alfriend et al [1], Schaub et al.[2], Schaub [3], Schaefer [4], Sengupta [5]). Specific insight on formation geometry is needed for mission planning and reconfiguration of formation dynamics and control. As stipulated by Yeh and Sparks [6], closed paths of relative motion traced out by a spacecraft under force-free motion permitted by the law of physics can be defined by Hill's equations. These are known as “legal formations” which satisfy the Hill-Clohessy Wiltshire (HCW) equations and must lie on the intersection of a plane and an elliptic cylinder with an eccentricity of $3/2$ in a moving coordinate system fixed to the chief spacecraft in the Local-Vertical-Local-Horizon (LVLH) frame.

Xiang and Jørgensen[7] have distinguished satellite formation flight compared to constellation if the relative position and relative velocity between the satellites in formation flight are controlled, and their relative altitudes can be controlled at certain parts. In many proposed missions, satellites are required to form a circular aperture in the plane perpendicular to the line-of-sight. For the optimal utilization of formation flight of micro-satellites, their relative motion dynamics and control under the existence of gravitational and environmental perturbations should be well taken into considerations, to

avoid their relative motion to keep changing and unstable. The main advantage of a LEO constellation over less complex, higher-altitude systems with fewer satellites is that the limited available frequencies that are useful for communicating through the atmosphere can be reused across the Earth's surface in an increased number of separated areas, or spotbeams, within each satellite's coverage footprint (Wood [8]). In addition, some countries that are geographically located straddling or near the Earth's equator may have some interests in Near Equatorial orbits for their development (Djojodihardjo and Harithuddin [9], Djojodihardjo and Zhahir [10]). In the well known HCW equation, the Earth is considered to be a point mass. However, since the Earth gravitational potential can be better represented by spheroidal (Vinti, [11], Djojodihardjo [12], Djojodihardjo and Kadarisman [13] or other harmonics (Alfriend et al, [1]), more accurate solution of legal formations should incorporate such gravitational potential, as well as other relevant disturbances. Without considering other disturbances, the dominant Earth's oblateness parameter, J_2 , for both the chief and the deputy satellite is here incorporated using linearized analysis, producing analytical solutions similar to that of the HCW equations. Such J_2 -Modified Hill's Equations describe the mean motion changes in both the in-plane and out-of-plane motion more accurately. By considering Near-Equatorial, the influence of J_2 may not vary significantly as compared to other inclined orbits. In this conjunction the objective of the present work is to obtain an assessment of the influence of J_2 on the formation flight orbit of twin satellites in near-Earth near-Equatorial orbit utilizing linearized J_2 modified Hill-Clohessy-Wiltshire Equation, and at the same time developing an in-house computational code for further development to include other higher approximations. It is with such motivation that the present work review and reassess the influence of J_2 perturbation on the formation flight of micro-satellites in Near Equatorial and Low Earth Orbits.

After establishing the foundation of Linearized Dynamics of the baseline HCW equations in sections II to V systematically, the J_2 Gravitational Perturbation Effects are incorporated to arrive at the Modified HCW Equation in VI and VII followed by validation, results and conclusions.

II. COORDINATE SYSTEMS AND TRANSFORMATIONS

The essential governing equation of spacecraft formation flight will be established by considering and identifying various coordinate systems. These coordinate systems can be defined by referring to Figure 1. The subscript N denotes a vector in the Earth Centered Inertial (ECI) frame, and a subscript O denotes a vector in the satellite-centered frame.

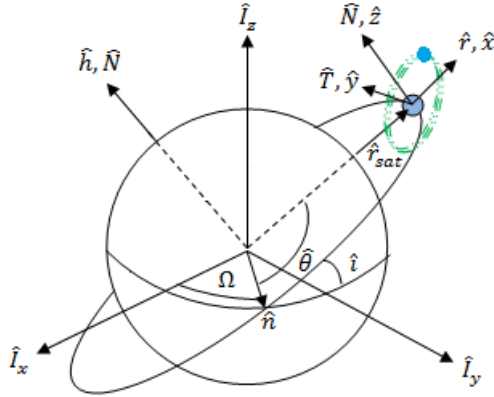


Figure 1. The ($r-\theta-i$) coordinate system used to describe the Chief and Deputy Satellite motion and the disturbance of J_2 in the local ($x-y-z$) coordinate system.

The ($r-\theta-i$) coordinate system (or Earth Centered Chief Satellite Orbital Plane coordinate system) is used in describing the J_2 disturbance in the local ($x-y-z$) coordinate system. The coordinate system elements r and the two Euler angles, θ and i , belong to the associated geometry for the transformation from the ECI frame to the ($r-\theta-i$) frame, utilizing the direction cosine matrix formed by the 3-1-3 Euler angle sets Ω , i and θ . These variables are known as the longitude of ascending node, the argument of latitude, and the angle of inclination, respectively. A similar direction cosine matrix (DCM) can be written in terms of the LVLH coordinate system in the ECI frame as expressed by (1) or (2):

$$[ON] = \begin{bmatrix} \cos \Omega \cos \theta - \sin \Omega \sin \theta \cos i & \sin \Omega \cos \theta + \cos \Omega \sin \theta \cos i & \sin \theta \sin i \\ -\cos \Omega \sin \theta - \sin \Omega \cos \theta \cos i & -\sin \Omega \sin \theta + \cos \Omega \cos \theta \cos i & \cos \theta \sin i \\ \sin \Omega \sin i & -\cos \Omega \sin i & \cos i \end{bmatrix} \quad (1)$$

$$[ON] = \begin{bmatrix} e_{rX} & e_{rY} & e_{rZ} \\ e_{\theta X} & e_{\theta Y} & e_{\theta Z} \\ e_{iX} & e_{iY} & e_{iZ} \end{bmatrix} \quad (2)$$

This is a direct rotation from ECI coordinates into the satellite-centered frame. Therefore, these two rotations are equivalent.

III. DEVELOPMENT OF RELATIVE DYNAMICS LINEAR MODEL

The equations of motion of the *deputy* spacecraft relative to the *chief* spacecraft is established following closely that of Djojodihardjo and Harithuddin [9], Alfriend et al [14], Djojodihardjo and Gunther [15]. Figure 2 exhibits the two spacecrafts orbiting the Earth.

The inertial position vector of the *chief* is \mathbf{R} , and that of the *deputy* is \mathbf{r} . The position vector of the *deputy* relative to the *chief* is \mathbf{p} , such that

$$\mathbf{r} = \mathbf{R} + \mathbf{p} \quad \text{or} \quad \mathbf{r}_d = \mathbf{r}_c + \mathbf{p} \quad (3)$$

One of the assumptions that should be made at this stage is that the relative distance between *chief* and *deputy* is small compared to the magnitude of \mathbf{R} , e.g. $\frac{\rho}{R} \ll 1$.

Following Newton's Gravitational Law, the equation of motion of an earth-orbiting body is.

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r} \quad ; \quad r \equiv \|\mathbf{r}\| \Rightarrow$$

$$\ddot{\mathbf{r}}_d = -\frac{\mu}{r_d^3} \mathbf{r}_d \quad ; \quad r_d \equiv \|\mathbf{r}_d\| \quad (4a)$$

as well as

$$\ddot{\mathbf{R}} = -\frac{\mu}{R^3} \mathbf{R} \quad ; \quad R \equiv \|\mathbf{R}\| \Rightarrow$$

$$\ddot{\mathbf{r}}_c = -\frac{\mu}{r_c^3} \mathbf{r}_c \quad ; \quad r_c \equiv \|\mathbf{r}_c\| \quad (4b)$$

Here μ is the standard gravitational constant of the earth, which is $3986 \text{ km}^3/\text{sec}^2$. In what follows, all perturbation components (derived from propulsive force, J_2 -perturbation, aerodynamics drag or third-body forces) will be ignored at the present stage. The vectors are all t -dependent. The equation of motion for the *deputy* in the moving frame can be further elaborated by substituting (3) into (4) to obtain the equation of motion of the *deputy* satellite.

Hence:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \ddot{\mathbf{p}} = -\frac{\mu}{\|\mathbf{R} + \mathbf{p}\|^3} (\mathbf{R} + \mathbf{p}) \quad (5)$$

and, subsequently

$$\ddot{\mathbf{p}} = -\ddot{\mathbf{R}} - \frac{\mu}{\|\mathbf{R} + \mathbf{p}\|^3} (\mathbf{R} + \mathbf{p}) \quad (6)$$

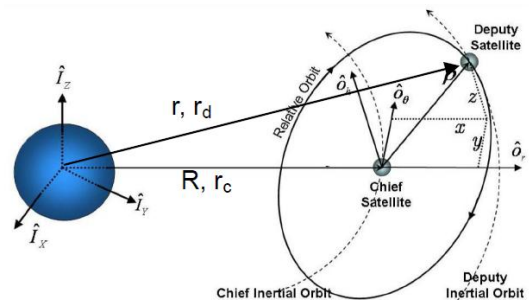


Figure 2: Coordinate System for defining relative motion (the figure is synthesized from Djojodihardjo and Harithuddin [8] and Ginn [16])

For circular orbit, $\omega = \sqrt{\frac{\mu}{R^3}}$, which represents the angular rate of the circular orbit around the center of Earth in the orbital plane. Here R is the semi-major axis or radius of the circular orbit and \mathbf{p} is the position vector of the *deputy* spacecraft in the relative (moving, orbiting) frame around the chief spacecraft. Accordingly:

$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{\mu}{R^3}} \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{p} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} \quad (7)$$

For the right-hand side of (6), one can expand $\|\mathbf{R} + \mathbf{p}\|^{-3}$ in a Taylor series approximation. Taylor series expansion about $F(\mathbf{R}, \mathbf{p}) = \|\mathbf{R} + \mathbf{p}\|^{-3}$ yields

$$\begin{aligned} \|\mathbf{R} + \mathbf{p}\|^{-3} &= [(\mathbf{R} + \mathbf{p}) \cdot (\mathbf{R} + \mathbf{p})]^{-\frac{3}{2}} \\ &= [(\mathbf{R} \cdot \mathbf{R}) + 2(\mathbf{R} \cdot \mathbf{p}) + (\mathbf{p} \cdot \mathbf{p})]^{-\frac{3}{2}} \\ &= \frac{1}{R^3} - \frac{3x}{R^4} + \frac{6x^2}{R^5} - \frac{3y^2}{2R^5} - \frac{3z^2}{2R^5} - \frac{10x^3}{R^6} + \frac{15xy^2}{2R^6} + \frac{15xz^2}{2R^6} + \dots \end{aligned} \quad (8)$$

Substituting (8) into (6) one obtains

$$\ddot{\mathbf{p}} = -\ddot{\mathbf{R}} - \mu \left\{ \begin{aligned} &\frac{1}{R^3} - \frac{3x}{R^4} + \frac{6x^2}{R^5} - \frac{3y^2}{2R^5} - \frac{3z^2}{2R^5} \\ &\quad - \frac{10x^3}{R^6} + \frac{15xy^2}{2R^6} + \frac{15xz^2}{2R^6} + \dots \end{aligned} \right\} (\mathbf{R} + \mathbf{p}) \quad (9)$$

Neglecting the terms with order higher than one, (13) becomes

$$\ddot{\mathbf{p}} = -\ddot{\mathbf{R}} - \frac{\mu \mathbf{R}}{R^3} - \frac{\mu}{R^3} \left(\mathbf{p} - \frac{3}{R^2} (\mathbf{R} \cdot \mathbf{p}) \mathbf{R} \right) \quad (10)$$

Substituting the equation of motion of the *chief* satellite

$$\ddot{\mathbf{R}} = -\frac{\mu}{R^3} \mathbf{R} \quad \text{into (10), one finally obtains:}$$

$$\ddot{\mathbf{p}} = \frac{\mu}{R^3} \left(\mathbf{p} - \frac{3}{R^2} (\mathbf{R} \cdot \mathbf{p}) \mathbf{R} \right) \quad (11)$$

(11) yields the desired relation for $\ddot{\mathbf{p}}$ in the inertial frame I . One needs to represent \mathbf{p} in the relative frame, R , around the chief spacecraft. One can write $\ddot{\mathbf{p}}_I$ in the inertial frame as,

$$\ddot{\mathbf{p}}_I = \ddot{\mathbf{p}}_R + 2(\boldsymbol{\omega} \times \dot{\mathbf{p}}_R) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}_R) + \dot{\boldsymbol{\omega}} \times \mathbf{p}_R \quad (12)$$

Thus, $\ddot{\mathbf{p}}$ in the relative frame,

$$\ddot{\mathbf{p}}_R = \ddot{\mathbf{p}}_I - 2(\boldsymbol{\omega} \times \dot{\mathbf{p}}_R) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}_R) - \dot{\boldsymbol{\omega}} \times \mathbf{p}_R \quad (13)$$

Substituting (11) into (17) and keeping only the linear terms, one obtain the kinematic relationship

$$\begin{aligned} \ddot{\mathbf{p}} &= (-\omega^2 \delta x - 2\omega \delta \dot{y} + \delta \ddot{x}) \mathbf{i} \\ &\quad + (-\omega^2 \delta y - 2\omega \delta \dot{x} + \delta \ddot{y}) \mathbf{j} + (\delta \ddot{z}) \mathbf{k} \end{aligned} \quad (14)$$

Substituting (11) into (14) yields the equation of motion

$$\ddot{\mathbf{p}} = -\omega^2 \left(\delta x \mathbf{i} + \delta y \mathbf{j} + \delta z \mathbf{k} - \frac{3}{R^2} (R \delta x) R \mathbf{i} \right) \quad (15)$$

Combining the kinematic relationship (14) with the equation of motion (15) yields

$$\begin{aligned} &(\delta \ddot{x} - 2\omega \delta \dot{y} - 3\omega^2 \delta x) \mathbf{i} \\ &\quad + (\delta \ddot{y} + 2\omega \delta \dot{x}) \mathbf{j} + (\delta \ddot{z} + \omega^2 \delta z) \mathbf{k} = 0 \end{aligned} \quad (16)$$

Hence, (16) gives the linearized Clohessy-Wiltshire equation:

$$\delta \ddot{x} - 2\omega \delta \dot{y} - 3\omega^2 \delta x = 0 \quad (17a)$$

$$\delta \ddot{y} + 2\omega \delta \dot{x} = 0 \quad (17b)$$

$$\delta \ddot{z} + \omega^2 \delta z = 0 \quad (17c)$$

These equations refer to the moving frame of reference in which they were derived. This moving frame is sometimes called CW-frame or Hill's frame. One advantage of the HCW equations is that the in-plane orbital motion (δx and δy directions) is uncoupled from the out-of-plane orbital motion (δz direction). In the present HCW equations, the following assumptions are made:

1. Eccentricity of the chief orbit is zero (circular), $e=0$
2. The angular rate is constant, $\dot{\omega}=0$
3. R is constant (circular orbit)

The homogeneous solution of the HCW equations derivation is carried out as follows. Define $\delta \mathbf{X} = [\delta x \ \delta y \ \delta z]^T$ and $\delta \mathbf{V} = [\delta \dot{x} \ \delta \dot{y} \ \delta \dot{z}]^T$. A subscript 0 denotes the initial condition. Then the solution of the linearized HCW equations can be represented in the following matrix form:

$$\delta \mathbf{X}(t) = [\Phi_{xx}] \delta \mathbf{X}_0 + [\Phi_{xv}] \delta \mathbf{V}_0 \quad (18)$$

where,

$$\Phi_{xx} = \begin{bmatrix} 4 - 3 \cos \omega t & 0 & 0 \\ 6 \sin \omega t - 6 \omega t & 1 & 0 \\ 0 & 0 & \cos \omega t \end{bmatrix}; \quad (19)$$

$$\Phi_{xv} = \begin{bmatrix} \sin \omega t / \omega & 2(1 - \cos \omega t) & 0 \\ 2(-1 + \cos \omega t) / \omega & 4 \sin \omega t / \omega - 3t & 0 \\ 0 & 0 & \sin \omega t / \omega \end{bmatrix}$$

and,

$$\delta \mathbf{V}(t) = [\Phi_{vx}] \delta \mathbf{X}_0 + [\Phi_{vv}] \delta \mathbf{V}_0 \quad (20)$$

where,

$$\Phi_{vx} = \begin{bmatrix} 3\omega \sin \omega t & 0 & 0 \\ 6\omega(1 + \cos \omega t) & 0 & 0 \\ 0 & 0 & -\omega \sin \omega t \end{bmatrix}; \quad (21)$$

$$\Phi_{vv} = \begin{bmatrix} \cos \omega t & 2(1 - \cos \omega t) & 0 \\ -2 \sin \omega t & -3 + 4 \cos \omega t & 0 \\ 0 & 0 & \cos \omega t \end{bmatrix}$$

(18)-(21) then describe the homogeneous solution of the Hill-Clohessey-Wiltshire (HCW) equation, which determines the position of the deputy spacecraft relative to the chief spacecraft as a function of t subject to initial conditions $\delta\mathbf{X}_0$ and $\delta\mathbf{V}_0$.

IV. BASELINE HILL-CLOHESSY-WILTSHIRE EQUATION

For a point mass or uniformly distributed sphere, the gravitational potential is

$$\mu = GM_e = 3.986005 \times 10^{14} \text{ m}^3 / \text{s}^2 \quad (22)$$

which is the first term of the more general Earth's gravitational potential. If J_2 is included, we have (Djojodihardjo [12], Alfried et al [14], Ginn [16], Anderson [17], and Schweighart [18])

$$U = -\frac{\mu}{\rho} + \frac{\mu R_e^2 J_2}{\rho^3} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \quad (23)$$

Baseline Hill-Clohessey Wiltshire Equations, for circular orbit around the Earth as the central body, assumed the Earth as point mass centered at its center of mass and the center of the orbit.

The equations of motion in the chief LVLH frame.

$$\frac{d^2 x}{dt^2} - 2\omega \frac{dy}{dt} - 3\omega^2 x = 0 \quad (24a)$$

$$\frac{d^2 y}{dt^2} + 2\omega \frac{dx}{dt} = 0 \quad (24b)$$

$$\frac{d^2 z}{dt^2} + \omega^2 z = 0 \quad (24c)$$

which is also known as the unperturbed or baseline HCW Equations. The angular velocity ω is given by

$$\omega = \sqrt{\frac{G(M+m)}{r^3}} = \sqrt{\frac{\mu}{r^3}} \quad (25)$$

The out-of-plane motion is modeled as a harmonic oscillator, where the in-plane motion is described as coupled harmonic oscillators. These second-order differential equations have the general solutions

$$x(t) = A \cos(nt + \alpha) + x_{\text{off}} \quad (26a)$$

$$y(t) = -2A \sin(nt + \alpha) - \frac{3}{2} n x_{\text{off}} t + y_{\text{off}}$$

$$(26b) \quad z(t) = B \cos(nt + \beta)$$

$$(26c)$$

where A , α , x_{off} , y_{off} , B and β are the six integral constants. The velocities are found as the time derivatives of (30a,b and c). In order to produce bounded relative motion, the radial offset term must be equal to zero to eliminate the secular growth present in the along-track direction. Setting the in-track offset term to zero, the bounded equations now have the form given by (26a,b and c). For the z direction, integration of: $\delta z(t) = B_0 \sin(\omega t + \alpha)$

$$(27)$$

yields:

$$z(t) = B \cos(nt + \alpha) + D_0 \quad (28)$$

Following Djojodihardjo and Harithuddin [9], the analytical solutions of the homogeneous HCW equations are obtained as follows. Define $\mathbf{X} = [x \ y \ z]^T$ and $\mathbf{V} = [\dot{x} \ \dot{y} \ \dot{z}]^T$. A subscript 0 denotes the initial condition. Then the solution of the linearized Clohessey-Wiltshire (HCW) equations can be represented in the following matrix form:

$$\mathbf{X}(t) = \Phi_{xx}(t) \mathbf{X}(t_0) + \Phi_{xv}(t) \mathbf{V}(t_0) \quad (29a)$$

$$\mathbf{V}(t) = \Phi_{vx}(t) \mathbf{X}(t_0) + \Phi_{vv}(t) \mathbf{V}(t_0) \quad (29b)$$

where $\Phi_{xx}(t)$, $\Phi_{xv}(t)$, $\Phi_{vx}(t)$ and $\Phi_{vv}(t)$ are state-transition matrices defined as in (19) - (21). The homogeneous solutions of the HCW equation determine the position and the velocity of the deputy spacecraft relative to the chief spacecraft as a function of t subject to initial conditions \mathbf{X}_0 and \mathbf{V}_0 .

V. RELATIVE BOUNDED MOTION

In formation flying, the motion of *deputy* satellite must remain bounded with respect to the chief satellite such that it experiences no secular drift and the formation configuration is maintained. One needs to find the condition such that the solutions of the Clohessey-Wiltshire equations are bounded [9][16]. (26a) and (26b) are coupled and they can be solved in parallel. Integrating (26b) yields an expression for $\dot{y}(t)$:

$$\dot{y}(t) = -2\omega x(t) + 2\omega x_0 + \dot{y}_0 \quad (30)$$

If one integrates (30) from 0 to t , one finds terms that grow unboundedly over time, namely the terms $2\omega x_0(t)$ and $\dot{y}_0(t)$. However, $y(t)$ can be made bounded and periodic given the condition

$$2\omega x_0 + \dot{y}_0 = 0 \quad (31)$$

Then, the solution for the in plane motion of the deputy satellite is:

$$x(t) = A_0 \sin(\omega t + \alpha) \quad (32a)$$

$$y(t) = 2A_0 \cos(\omega t + \alpha) + C_0 \quad (32b)$$

where A_0 , phase angle α and integration constant C_0 depend on the initial conditions. The out-of-plane motion is decoupled from the in-plane motion and its solution takes on the form of a simple harmonic oscillator:

$$z(t) = B_0 \sin(\omega t + \alpha) \quad (32c)$$

where the amplitude B_0 and the phase angle α are constants which depend on the initial conditions. The out-of-plane motion is periodic and bounded with respect to the chief satellite.

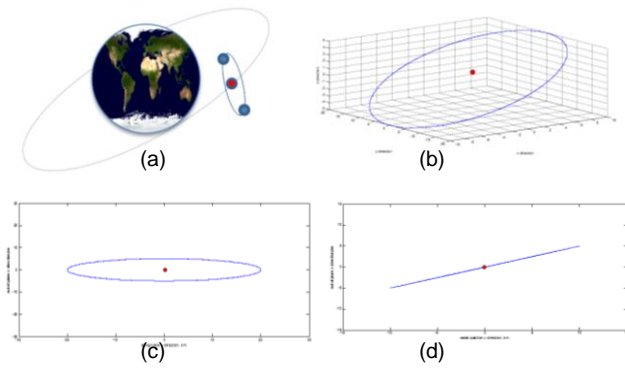


Figure 3.(a) Sketch of projected circular orbit in inertial frame; (b)-(d) Illustration of the relative position of deputy satellite in relative frame centered on the chief satellite.

$$U = \frac{\mu}{r} \left[1 - \sum_{l=2}^{\infty} J_l \left(\frac{R_{\oplus}}{r} \right)^l P_l \left[\cos(\phi_{gc_{sat}}) \right] + \sum_{l=2}^{\infty} \sum_{m=1}^l \left(\frac{R_{\oplus}}{r} \right)^l P_{l,m} \left[\cos(\phi_{gc_{sat}}) \right] \{ C_{l,m} \cos(m\lambda_{sat}) + S_{l,m} \sin(m\lambda_{sat}) \} \right] \quad (34)$$

The solutions (32) define a family of bounded trajectories for the deputy satellite with periodic motion in the relative frame under the assumptions of the baseline HCW-equations. The motion of the deputy satellite, if projected onto the y-z plane, follows an ellipse of semi-major axis $2A_0$ and semi-minor axis A_0 . Figures 3 (b) to (d) exhibit the geometry of the relative position of the deputy satellite in a relative frame centered on the chief satellite, while Figs. 3(a) illustrates the motion of the deputy satellite with respect to the chief satellite as a projected circular orbit in inertial frame.

VI. LINEARIZATION OF THE INFLUENCE OF J_2 ON THE GRAVITATIONAL POTENTIAL

The main gravitational perturbation effect is due to J_2 , the equatorial bulge term. The J_2 term changes the orbit period, a drift in perigee, a nodal precession rate and periodic variations in all the elements. In what follows, the **right ascension rate** which is equal to [14]

$$\dot{\Omega} = -\frac{3}{2} J_2 \left(\frac{R_E}{p} \right)^2 n \cos i \quad (33)$$

is considered. Any non-spherical body can be modeled using spherical harmonics, which can then be differentiated into three types of harmonics, i.e. the zonal, sectorial, and tesseral ones. For the Earth, J_2 represents the zonal harmonic, i.e. the equatorial bulge and is the largest coefficient of the Earth's gravitational potential. The difference in equatorial and polar radii is mainly due to this bulge, which is about 21km. Various reference frames that are required to describe the motion of a satellite in orbit around the Earth. These include the geometry used to describe the potential due to J_2 . For an orbit around the Earth of about 800 km altitude, the J_2 effect is much larger in comparison with other perturbations such as atmospheric drag, solar radiation pressure and electro-magnetic effects [1] [14][19].

6.1. Adding the J_2 Perturbation

Considering the Earth as a spherical central body of uniform density in the earlier section, the two-body equations of motion can be written in a simple form. However, the Earth is a non-spherical mass of finite size and imposes a gravitational potential due to an aspherical central body. More accurate two-body equations of motion can be made by considering and determining the gravitational potential due to an aspherical central body, using a coordinate system depicted in Figure 4 to describe the aspherical gravitational potential. The potential that describes an aspherical central body is given by (Vinti [11], Djojodihardjo [12], Djojodihardjo and Kadarusman [13], Anderson [17], Tapley et al [20]):

where J_l , $C_{l,m}$, and $S_{l,m}$ are gravitational coefficients and R_{\oplus} is the equatorial radius of the Earth. The first term is the two-body potential, whereas the second term is the potential due to zonal harmonics.

An aspherical body which only deviates from a perfect sphere due to zonal harmonics is axially symmetric about the Z-axis. The third term represents two other harmonics. The sectorial harmonics, where $l = m$, represent bands of longitude, and tesseral harmonics, where $l \neq m \neq 0$, represent tile-like regions of the Earth.

The J_2 coefficient is about 1000 times larger than the next largest aspherical coefficient, and is therefore very important when describing the motion of a satellite around the Earth. The potential due to the J_2 disturbance can be obtained from Vinti [11] as

$$U_{zonal} = \frac{\mu}{r} J_2 \left(\frac{R_{\oplus}}{r} \right)^2 P_2 \left[\cos(\phi_{gc_{sat}}) \right] \quad (35a)$$

which can further be reduced to

$$U = -\frac{\mu}{\rho} + \frac{\mu R_E^2 J_2}{\rho^3} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \quad (35b)$$

where $P_2 \left[\cos(\phi_{gc_{sat}}) \right]$ is the associated Legendre

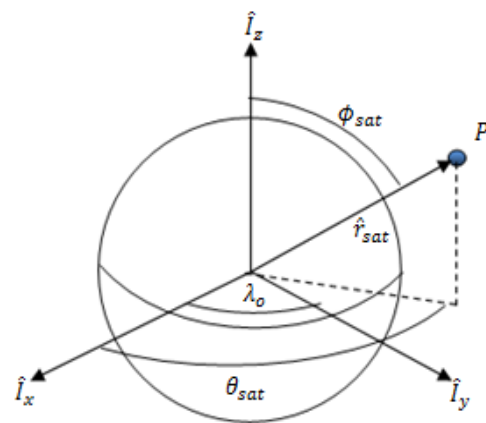


Figure 4. Geometry used to derive the gravitational potential

polynomial of J_2 ; the second zonal gravitational coefficient according to the JGM -2 model [16][20] has been calculated as $J_2 = 1.082626925638815 \times 10^{-3}$. The co-latitude may be written as

$$\sin^2(\phi_{gc_{sat}}) = 1 - \frac{Z^2}{r^2} \quad (36)$$

The acceleration due to J_2 in the **ECI frame** is then calculated as the gradient of the potential

$$\nabla U_{J_2} = \bar{J}_2 = \begin{bmatrix} \frac{\partial U_{J_2}}{\partial X} \\ \frac{\partial U_{J_2}}{\partial Y} \\ \frac{\partial U_{J_2}}{\partial Z} \end{bmatrix} = -\frac{3\mu J_2 R_\oplus^2}{2r^5} \begin{bmatrix} X \left(1 - \frac{5Z^2}{r^2}\right) \\ Y \left(1 - \frac{5Z^2}{r^2}\right) \\ Z \left(3 - \frac{5Z^2}{r^2}\right) \end{bmatrix} \quad (37)$$

The chief and deputy equations of motion can be rewritten in the inertial frame as

$$\ddot{\bar{r}}_c = -\frac{\mu}{r_c^3} \bar{r}_c + \bar{J}_{2c} \quad (38)$$

$$\ddot{\bar{r}}_d = -\frac{\mu}{r_d^3} \bar{r}_d + \bar{J}_{2d} \quad (39)$$

The acceleration due to J_2 in the LVLH frame may be calculated from the gradient in the r and Z directions:

$$\begin{aligned} \nabla U_{J_2} &= \frac{\partial U_{J_2}}{\partial r} \hat{e}_r + \frac{\partial U_{J_2}}{\partial z} \hat{e}_z \\ &= -\mu J_2 R_\oplus^2 \left[\left(\frac{3}{2r^4} - \frac{15z^3}{2r^6} \right) \hat{e}_r + \frac{3z}{r^5} \hat{e}_z \right] \end{aligned} \quad (40)$$

where the Z component may be expressed in the LVLH frame as [5][16][18]

$$\begin{aligned} \hat{e}_z &= \sin i \sin \theta \hat{e}_r + \sin i \cos \theta \hat{e}_\theta + \cos i \hat{e}_N \\ z &= r \cos \phi = r \sin i \sin \theta \end{aligned} \quad (41)$$

Substituting this equation into (40), one obtained the acceleration due to J_2 to be

$$\begin{aligned} \nabla U_{J_2} &= \frac{\partial U}{\partial r} \hat{e}_r + \frac{\partial U}{\partial z} \hat{e}_z \\ &= -\mu J_2 R_\oplus^2 \left[\left(\frac{3}{2r^4} - \frac{15z^3}{2r^6} \right) \hat{e}_r + \frac{3z}{r^5} \hat{e}_z \right] \end{aligned} \quad (42a)$$

or

$$\begin{aligned} \nabla U_{J_2} = \bar{J}_2 &= -\frac{\mu J_2 R_\oplus^2}{r^4} \begin{bmatrix} \left(\frac{1}{2} - \frac{3 \sin^2 i \sin^2 \theta}{2} \right) \\ \sin^2 i \sin \theta \cos \theta \\ \sin i \cos i \sin \theta \end{bmatrix} \\ &= -\frac{\mu J_2 R_\oplus^2}{r^4} \begin{bmatrix} \left(\frac{1}{2} - \frac{3 \sin^2 i \sin^2 \theta}{2} \right) \hat{i} \\ + \sin^2 i \sin \theta \cos \theta \hat{j} \\ + \sin i \cos i \sin \theta \hat{k} \end{bmatrix} \end{aligned} \quad (42b)$$

in Earth-Centered Inertial (ECI) frame of reference.

The chief and deputy equations of motion in the inertial frame due to J_2 in the ECI frame is given by (38) and (39). The linearized equations of motion for the chief and deputy satellites in ECI is given by

$$\ddot{\rho} = \frac{\mu}{R^3} \left(\rho - \frac{3}{R^2} (\mathbf{R} \cdot \rho) \mathbf{R} \right) \rightarrow \ddot{\rho} = \frac{\mu}{r_c^3} \left(\rho - \frac{3}{r_c^2} (\mathbf{r}_c \cdot \rho) \mathbf{r}_c \right) \quad (43)$$

The inertial relative position and velocity is defined as the position and velocity of the deputy relative to the chief.

$$[\bar{\rho}]_N = \bar{r}_d - \bar{r}_c \Rightarrow \rho = r_d - r_c \quad (44)$$

$$[\dot{\bar{\rho}}]_N = \dot{r}_d - \dot{r}_c \Rightarrow \dot{\rho} = \dot{r}_d - \dot{r}_c \quad (45)$$

$$[\ddot{\bar{\rho}}]_N = \ddot{r}_d - \ddot{r}_c \Rightarrow \ddot{\rho} = \ddot{r}_d - \ddot{r}_c \quad (46)$$

$$\omega = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{\mu}{R^3}} \end{bmatrix}; \quad r_c = \begin{bmatrix} r_c \\ 0 \\ 0 \end{bmatrix}; \quad \rho = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}; \quad r_d = \begin{bmatrix} r_c + \delta \\ \delta y \\ \delta z \end{bmatrix} \quad (47)$$

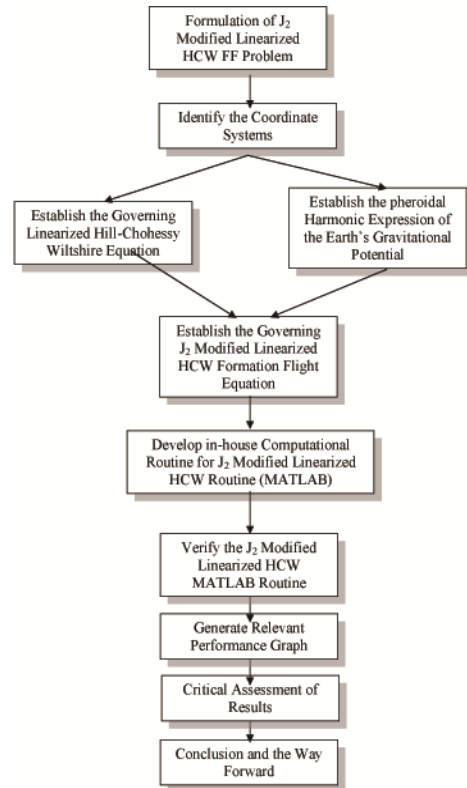


Figure 5. Overall methodology following the present approach

Hence the components in **ECI** Frame of reference is given by

$$\mathbf{r}_c = \begin{bmatrix} r_c \sin \theta \cos i \\ r_c \cos \theta \sin \Omega \\ r_c \sin \theta \sin i \end{bmatrix}; \quad \mathbf{r}_d = \begin{bmatrix} r_d \sin \theta_d \cos i_d \\ r_d \cos \theta_d \sin \Omega \\ r_d \sin \theta_d \sin i_d \end{bmatrix} \quad (48)$$

Similar to unperturbed HCW case, in LVLH, the solution of the equations of motion can be represented in the following matrix form:

$$\delta \mathbf{X}(t) = [\Phi_{XX}] \delta \mathbf{X}_0 + [\Phi_{XV}] \delta \mathbf{V}_0 \quad (49)$$

where appropriate terms like below have to be formulated,

$$\text{and } \delta \mathbf{V}(t) = [\Phi_{VX}] \delta \mathbf{X}_0 + [\Phi_{VV}] \delta \mathbf{V}_0 \quad (50)$$

With the present baseline formulation, the approach follows closely a combination of Ginn's [16] and Schweighart's [18] linearized approach, the detail of which is given by Djojodihardjo and Tee [21][22]. Computational procedure and code are then developed following the scheme depicted in Figure 5.

VII. LINEARIZED J₂ MODIFIED HCW EQUATIONS

Proceeding with further algebraic manipulations, the governing equations for the calculation of the influence of J₂ on the linearized HCW Orbit are obtained and summarized below [16] [18][21][22].

$$x(t) = \left[\begin{array}{c} \frac{5s+3}{s-1} x_{h0} + \frac{2\sqrt{1+s}}{n(s-1)} \dot{y}_{h0} \\ + \frac{1}{4} \frac{A_{J2} (3k - 2n\sqrt{1-s}) \sin^2 i}{k(-n^2 + n^2 s + 4k^2)} \end{array} \right] \cos(n\sqrt{1-s}t) \quad (51)$$

$$y(t) = \left[\begin{array}{c} -\frac{1}{4} \frac{A_{J2} (3k - 2n\sqrt{1-s}) \sin^2 i}{k(-n^2 + n^2 s + 4k^2)} \cos 2kt \\ + \frac{\dot{x}_{h0}}{n\sqrt{1-s}} \sin(n\sqrt{1-s}t) \\ - \frac{4(s+1)}{s-1} x_{h0} - \frac{2\sqrt{1+s}}{n(s-1)} y_{h0} \\ \frac{2(5s+3)\sqrt{1+s}}{(1-s)^{\frac{3}{2}}} x_{h0} + \frac{4(1+s)}{n(1-s)^{\frac{3}{2}}} \dot{y}_{h0} \\ + \frac{1}{2} \frac{A_{J2} (2ns - 3k\sqrt{1+s} + 2n) \sin^2 i}{k\sqrt{1-s}(-n^2 + n^2 s + 4k^2)} \end{array} \right] \sin(n\sqrt{1-s}t) \quad (52)$$

$$z(t) = z_{h0} \cos(n\sqrt{1+3s}t) + \frac{\dot{z}_{h0}}{n\sqrt{1+3s}} \sin(n\sqrt{1+3s}t) \quad (53)$$

where

$$A_{J2} \equiv -3n^2 J_2 \frac{R_{\oplus}^2}{r_c} \quad (54)$$

These equations are incorporated in the in-house MATLAB computational routine.

VIII. EXAMPLES AND VALIDATION

A. Validation of Clohessy-Wiltshire Model

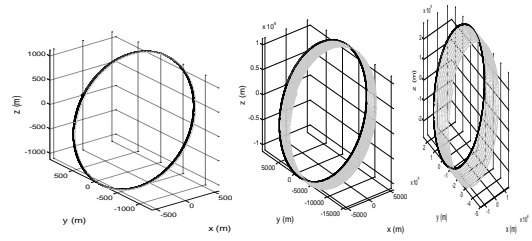


Figure 6. Relative trajectory comparison for $\rho = 1$ km, 10 km, 25 km, and 50 km with $e=0$ and $a= 7225$ km.

To demonstrate the relative satellite motion modeled by the Clohessy-Wiltshire equations, the projected circular orbit trajectory is simulated via MATLAB and worked out in [9][15] are shown partially for completeness and comprehensive impression, as exemplified in Figures 6 and 7. The trajectory follows the initial conditions defined by the set of solutions presented in Table 1.

TABLE I.
CHIEF'S ORBITAL ELEMENTS AND DEPUTY'S INITIAL CONDITIONS WITH RESPECT TO CHIEF'S

Chief Satellite	
Altitude, h (km)	847
Eccentricity, e	0
Orbit Inclination, I (deg)	10^0
Right Ascension of the Ascending Node, Ω (deg)	0^0
Argument of Perigee ω (deg)	0^0
Mean Anomaly at Epoch, M (deg)	0^0
Deputy Satellite Starting Condition (Chief-centered Frame)	
x_0 (km)	0.0
y_0 (km)	5.0
z_0 (km)	0.0
v_{x0} (km/s)	$0.5785 \cdot 10^{-3}$
v_{y0} (km/s)	0.0
v_{z0} (km/s)	$1.1570 \cdot 10^{-3}$
i (deg)	10
θ (deg)	nt

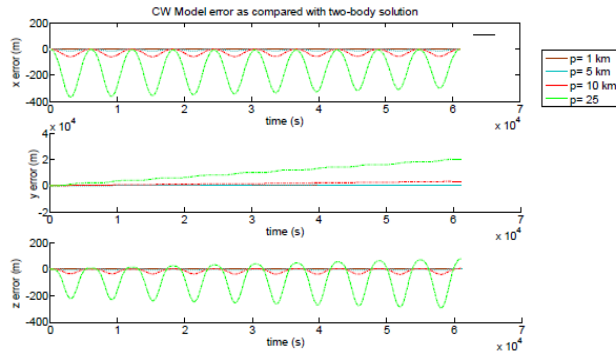


Figure 7: Clohessy-Wiltshire model error for $\rho = 1$ km, 5 km, 10 km, 25 km, and 50 km with $e=0$ and $a= 7225$ km.

B. Comparison and Validation of Baseline Clohessy-Wiltshire Model of Twin-Satellite Orbits with J_2 -Perturbed Ones

To demonstrate the influence of J_2 on the linearized (HCW) orbit of the Twin Satellite Formation Flying Orbits, the J_2 perturbed linearized HCW equations orbits are compared with the baseline ones. The initial condition are those given in Table 1. The results are exhibited in Figs. 8 to 12.

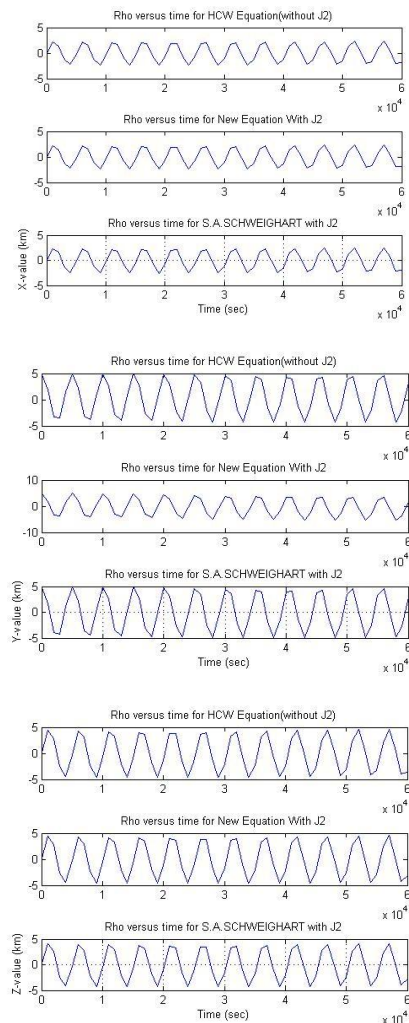


Figure 8: Comparison of the X,Y and Z values, respectively, of Deputy Satellite orbit around the Chief Satellite as the solution between baseline HCW, linearly modified HCW Equation and Schweighart's results (the latter two incorporate the effect of J_2).

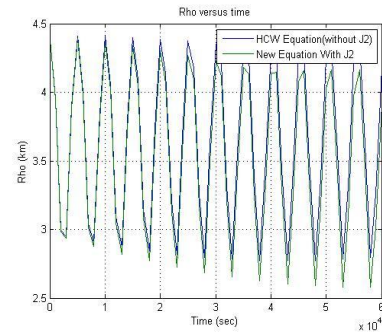


Figure 9: Comparison of the Deputy Satellite orbital radius around the Chief Satellite as the solution of Clohessy-Wiltshire Equation (without J_2) and incorporating the influence of J_2 , using linearized modified Clohessy-Wiltshire Equation.

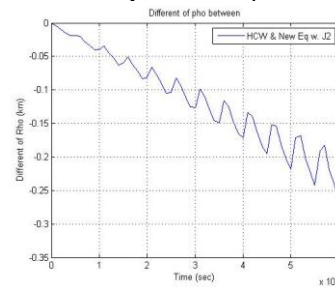


Figure 10: (a) The difference between the radius of the orbit of the Deputy Satellite around the Chief Satellite as the solution of the original linearized HCW Equation (without J_2) and that incorporating the influence of J_2 , using linearized J_2 modified HCW Equation

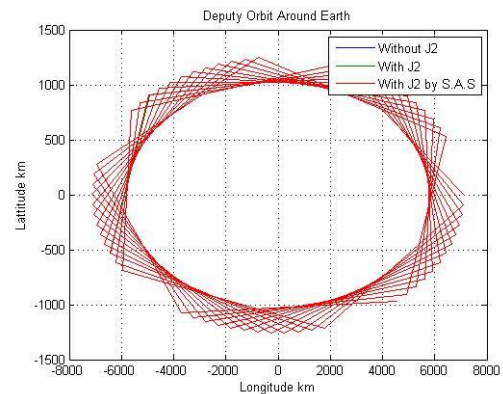


Figure 11: Comparison of Baseline Ground-Track of the Deputy and Chief Satellites orbits as the solution of baseline HCW and the Linearly Modified HCW equation which incorporate the influence of J_2 .

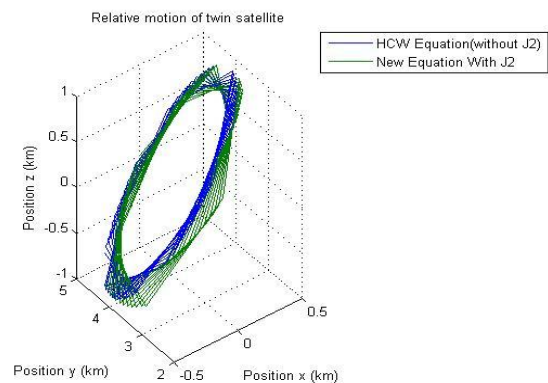


Figure 12: Comparison of Deputy Satellite orbit around the Chief Satellite as the solution of baseline Clohessy-Wiltshire Equations (without J_2) and Linearly Modified HCW equations which incorporate the influence of J_2 .

The results show that the equations derived in this work have close similarity with the ones derived by Schweighart, although quantitatively there are differences. It should be noted that Schweighart's solutions originate from different J_2 linearization compared to the present work. Such difference may be attributed to the notion that Schweighart's equations do not include the drift of the ascending node of a satellite under the influence of the J_2 disturbance.

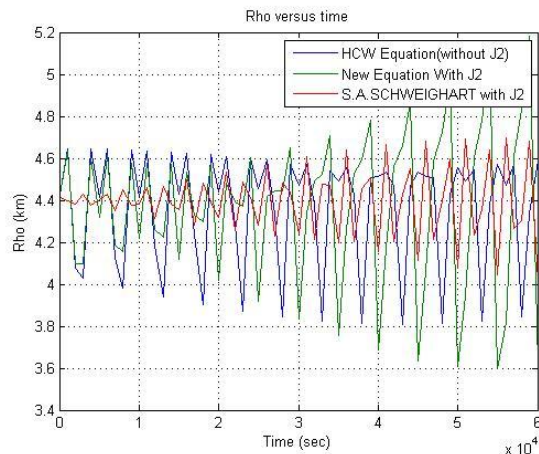


Figure 12: Comparison of the Deputy Satellite orbital radius around the Chief Satellite as the solution of the baseline Clohessy-Wiltshire Equations, the linearized J_2 modified HCW Equations and similar solution obtained by Schweighart [17].

IX. SUMMARY OF RESEARCH RESULTS

The work carried out in the present paper is summarized in Table 2.

TABLE II
SUMMARY OF RESEARCH RESULTS

J2 Modified Linearized Hill-Clohessy-Wiltshire Computational Routine for Twin Satellites Formation Flight	
1	Establishment of Governing Equation, as one of many possible linearization approach
2	Development of MATLAB based Computational Routine for the computation of J2 Modified Linearized Hill-Clohessy-Wiltshire Equation
3	Validation of Unperturbed Linearized Hill-Clohessy-Wiltshire Equation
4	Validation of J2 Modified Linearized Hill-Clohessy-Wiltshire Equation

X. CONCLUSIONS

Linearized Hill-Clohessy-Wiltshire equations have been utilized in developing modified form to take into account the influence of J_2 on the orbits of twin spacecraft in formation flight in near-Earth orbits. For Near Equatorial orbits the variation of J_2 is less apparent. Various relevant approaches and recent work on this issue have been synthesized into a novel and simplified

approach, capitalizing on the balance between linearized approach and expected fidelity of the obtained solution, as stipulated by many earlier work. Judging from the accuracy estimation of simplified linearized approach, the exhibited computational results were obtained using J_2 linearized HCW equation. The original (baseline) linearized HCW approach and linearized J_2 -modified HCW equation also exhibit the merit of simple analysis, which could be extended to incorporate other parameters. The relevance of parametric study as a preliminary step towards optimization efforts has been demonstrated in the presentation of the results. The computation that has been performed using in-house developed MATLAB program. As a particular example, for low earth orbit (i.e. 847 km), the error is about 0.25km from the desired relative position in the LVLH or Hill frame after 16.67 hours.

XI. ACKNOWLEDGMENT

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